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Expansion Theory for the Elliptic Motion of Arbitrary Eccentricity and Semi-major Axis

XIII. Developments of the Mean Anomaly g' of the Disturbing Body and cos mg` , sin mg` in the Sectorial Regularized Theory

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ABSTRACT. In this paper of the series, the fourth step of the sectorial regularized theory will start by establishing the exact literal analytical expressions for g` and the doubly trigonometric series representations of cos mg` and sin mg` where *m* positive integer. Moreover, some recurrence formulae are also established to facilitate digital computations for the coefficients of the series representations of cos mg` and sin mg`. All the formulations developed in the paper are general in the sense that they are valid whatever the types and the number of sectors forming the divisions situation of the elliptic orbit may be. In addition they are also valid during any revolution of the perturbed body in its Keplerian orbit. Finally, we include some numerical results for the coefficients of the trigonometric series representations of $dg/d\theta_j^{(i)}$ ($\theta_j^{(i)}$ are the sectorial variables) to provide test examples for constructing computational algorithms .

1. Introduction

The conventional methods of general perturbations of celestial mechanics do not yield manageable series solutions for some orbital systems, natural or artificial. In general these systems are characterized by a highly oscillating perturbing force resulting in divergent, or at best weak, convergent series solutions.

In a series of previous communications references[1-12], a regularization approach was introduced to regularize the highly oscillating functions, involved in the above orbital systems. Our approach based on the idea of orbit segmentation into sectors. By dividing the elliptic orbit into sectors – with a different variable for each sector (which we called a sectorial variable) – the highly oscillating perturbation function is then segmented into fewer oscillations (one for each sector) and every function could be developed in convergent series expansion. The aims and the developments of each paper of the series are summarized in the introduction of[12].

In the present paper, the expansions of coordinate functions of the disturbing body are considered by establishing the exact literal analytical expressions, for *g*` and for the coefficients of the doubly trigonometric series representations of cos mg` and sin mg` where *m* positive integer and *g*` the mean anomaly of the disturbing body whose orbit is considered to be whole elliptic orbit (without segmentation). Computational developments are also considered by establishing some recurrence formulae to facilitate the digital computations. Numerical results for the coefficients of the trigonometric series representation of $dg`/d\theta_j^{(i)}$ are included to provide test examples for constructing computational algorithms. All the formulation of the present paper are valid for any number and type of the sectors forming the divisions situation of the elliptic orbit of the perturbed body (always segmented) during any revolution. Before starting the analysis it is preferable to recall the representation equations for the general and arbitrary sectorial divisions of elliptic orbits as follows.

1.1. Representation Equations for Sectorial Divisions of Elliptic Orbits

Let an elliptic orbit of semi-major axis *a* and of eccentricity *e* be divided into $\mathbf{N}^{(i)}$; $i = 1,2,3,4$ elliptic sectors such that the $N^{(1)}$, $N^{(2)}$ sectors are positive and the $N^{(3)}$, $N^{(4)}$ are the negative elliptic sectors. (All elliptic sectors above the major axis are positive, while those below the major axis are negative elliptic sectors). Let these sectors be formed by the components $r_j^{(i)}$ of the four vectors $\mathbf{R}^{(i)}$ each of dimension $N^{(i)} + 1$ and the independent variables of these sectors be denoted by the components $\theta_j^{(i)}$ of the vector $\theta^{(i)}$ each of dimension $N^{(i)}$. Then the representation equations for these sectorial divisions of any elliptic orbit are given in accordance with Section 3 of[4] as

$$\sin\frac{u}{2} = \eta_{j+1}^{(1)}\{1 - l_j^{(1)}\cos^2\theta_j^{(1)}\};\cos\frac{u}{2} = +\sqrt{1-\sin^2\frac{u}{2}}$$

$$\forall 0^\circ \le \theta_j^{(1)} \le 90^\circ; j = 1,2,\ldots,N^{(1)}, \qquad (1.1)$$

$$\cos\frac{f}{2} = \eta_j^{(2)}\{1 - l_j^{(2)}\cos^2\theta_j^{(2)}\};\sin\frac{f}{2} = +\sqrt{1-\cos^2\frac{f}{2}}$$

$$\forall 90^o \le \theta_j^{(2)} \le 180^o; j = 1,2,...,N^{(2)}, \quad (1.2)$$

$$\cos\frac{f}{2} = -\eta_{j+1}^{(3)}\{1 - l_j^{(3)}\cos^2\theta_j^{(3)}\};\sin\frac{f}{2} = +\sqrt{1-\cos^2\frac{f}{2}}$$

$$\forall 180^o \le \theta_j^{(3)} \le 270^o; j = 1,2,...,N^{(3)}, \quad (1.3)$$

$$\sin\frac{u}{2} = \eta_j^{(4)}\{1 - l_j^{(4)}\cos^2\theta_j^{(4)}\};\cos\frac{u}{2} = -\sqrt{1-\sin^2\frac{u}{2}}$$

$$\forall 270^o \le \theta_j^{(4)} \le 360^o; j = 1,2,...,N^{(4)}, \quad (1.4)$$

$$;\ \eta_k^{(k_1)} = \left\{\frac{r_k^{(k_1)} - a(1-e)}{2ae}\right\}^{1/2} ;k_1 = 1 \text{ or } 4, \quad (1.5)$$

$$\eta_k^{(k_2)} = \left\{\frac{a(1+e) - r_k^{(k_2)}}{2er_k^{(k_2)}}(1-e)\right\}^{1/2} ;k_2 = 2 \text{ or } 3, \quad (1.6)$$

$$l_k^{(1)} = 1 - \left\{\frac{r_k^{(1)} - a(1-e)}{r_{k+1}^{(1)} - a(1-e)}\right\}^{1/2},\ 0 < l_k^{(1)} \le 1\ (l_1^{(1)} = 1), \quad (1.7)$$

$$l_k^{(2)} = 1 - \left\{\frac{a(1+e) - r_{k+1}^{(2)}\ r_k^{(2)}}{a(1+e) - r_k^{(2)}\ r_{k+1}^{(2)}}\right\}^{1/2},\ 0 < l_k^{(2)} \le 1\ (l_{N^{(2)}}^{(2)} = 1), \quad (1.8)$$

$$l_k^{(3)} = 1 - \left\{\frac{a(1+e) - r_k^{(3)}\ r_{k+1}^{(3)}}{a(1+e) - r_{k+1}^{(3)}\ r_k^{(3)}}\right\}^{1/2},\ 0 < l_k^{(3)} \le 1\ (l_1^{(3)} = 1), \quad (1.9)$$

$$l_k^{(4)} = 1 - \left\{\frac{r_{k+1}^{(4)} - a(1-e)}{r_k^{(4)} - a(1-e)}\right\}^{1/2},\ 0 < l_k^{(4)} \le 1\ (l_{N(4)}^{(4)} = 1), \quad (1.10)$$

u and f being the eccentric and the true anomalies.

2. Literal Analytical Expressions of g`

In this section, the literal analytical expressions for the Fourier expansion of the mean anomaly g` of the disturbing body will be established in terms of the sectorial variables $\theta_j^{(i)}$. Such expressions are general in the sense that they are valid whatever the types and the number of sectors forming the divisions situation of the elliptic orbit may be. The developments of this section will be covered through the following points.

2.1 Fourier Expansions of $dg`/d\theta_j^{(i)}$

The mean anomaly $g`$ of the disturbing body is defined in terms of its mean motion $n`$ by

$$\frac{dg`}{dt} = n` . \tag{2.1}$$

To express $g`$ in terms of the sectorial variable $\theta_j^{(i)}$, Equation (2.1) is to be written as

$$\frac{dg`}{d\theta_j^{(i)}} = v\,(n_c \frac{dt}{d\theta_j^{(i)}})\ ;\ i=1,2,3,4\ ;\ \ j=12,...,N^{(i)}, \tag{2.2}$$

where $v = n`/n_c$ is the ratio between the mean motions of the disturbing and perturbed bodies respectively.

According to Theorem 4 of[12], we can get for the Fourier expansions of $dg`/d\theta_j^{(i)}$ the forms

$$\frac{dg`}{d\theta_j^{(i)}} = v\sum_{s=1}^{\infty} A_{s,j,i}\ \sin 2s\theta_j^{(i)}\ \ ;\ \ \forall(i-1).\ 90^{o} \le \theta \le i.90^{o}, \tag{2.3}$$

$$;A_{s,j,1} \equiv H_s^{(1)}(\eta_{j+1}^{(1)}\ ,\ l_j^{(1)}\ ,\ e)\ ;\ \forall s\ge 1\ ;\ j=1,2,...,N^{(1)}, \tag{2.4}$$

$$A_{s,j,2} \equiv -(1-e^2)^{3/2} H_s^{(-2)}(\eta_j^{(2)},\ l_j^{(2)}\ ,\ e)\ ;\ \forall s\ge 1\ ;\ j=1,2,...,N^{(2)}, \tag{2.5}$$

$$A_{s,j,3} \equiv -(1-e^2)^{3/2} H_s^{(-2)}(\eta_{j+1}^{(3)},\ l_j^{(3)}\ ,\ e)\ ;\ \forall s\ge 1\ ;\ j=1,2,...,N^{(3)}, \tag{2.6}$$

$$A_{s,j,4} \equiv -H_s^{(1)}(\eta_j^{(4)}\ ,\ l_j^{(4)}\ ,\ e)\ ;\ \forall s\ge 1\ ;\ J=1,2,...,N^{(4)}, \tag{2.7}$$

where the $H`$s functions are given for the possible values of $\eta`s$, $l`s$ and e as

$$H_s^{(q)}(\varepsilon_1,\varepsilon_2,\xi) = \begin{cases} G_0^{(q)}(\varepsilon_1,\varepsilon_2,\xi) - G_2^{(q)}(\varepsilon_1,\varepsilon_2,\xi)\ ,\ \text{if}\ s=1 \\ \\ -G_{s+1}^{(q)}(\varepsilon_1,\varepsilon_2,\xi)\ ,\ \text{if}\ s\rangle 1\ ; \end{cases} \tag{2.8}$$

where for q = 1, for any real number $q \notin z^+$ [z^+ = { 0,1,2,...}] and the possible values of $\varepsilon_{1,2}$ and ξ, the $G`s$ functions are those given in Table 1 in which $< j >$

TABLE 1. The G`s functions for the possible values of their parameters.

Parameters	Literal analytical expressions of the G`s functions
$\forall s\geq 0;0\langle\varepsilon_{1,2}\big(\langle 1;0\langle\xi\langle 1;q\notin z^{+}$	$G_s^{(q)}(\varepsilon_1,\eta_2,\xi)=2\sum_{n=\langle\frac{s}{2}\rangle}^{\infty}\sum_{l=\langle\frac{n}{2}\rangle}^{\infty}\sum_{j=0}^{l}\sum_{i=j}^{\infty}(-1)^{n+l+i}\binom{4n}{2n-s}\binom{2l}{n}\binom{-\frac{1}{2}}{l-j}\binom{q}{i}\binom{i}{j}2^{j-4n}\xi^{i}\varepsilon_1^{2l+1}\varepsilon_2^{n-1}.$ (I – 1)
$\forall s\geq 0;0\langle\varepsilon_{1}\big(\langle 1;0\langle\xi\langle 1;\varepsilon_2=1;q\notin Z^{+}$	$G_s^{(q)}(\varepsilon_1\varepsilon_2,\xi)=2(-1)^{s}\sum_{l=\langle\frac{s}{2}\rangle}^{\infty}\sum_{j=0}^{l}\sum_{i=j}^{\infty}(-1)^{l+i}\binom{-\frac{1}{2}}{l-j}\binom{q}{j}\binom{i}{j}\binom{4l}{2l-s}\xi^{i}2^{j-4l}\varepsilon_1^{2l+1}.$ (I – 2)
$\forall s\geq 0;0\langle\varepsilon_{1,2}\ \langle 1;0\langle\xi\langle 1;q=1$	$G_0^{(q)}(\varepsilon_1,\varepsilon_2,\xi)=2\varepsilon_1\varepsilon_2(1-\xi)+\varepsilon_2\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}\binom{-\frac{1}{2}}{l-1}\varepsilon_1^{2l+1}\{2l(\xi+1)+\xi-1\}+\varepsilon_2\sum_{n=1}^{\infty}\sum_{l=\langle\frac{n}{2}\rangle}^{\infty}\frac{(-1)^{l+n-1}}{l}\binom{-\frac{1}{2}}{l-1}\binom{2l}{n}$ $\times\binom{4n}{2n}\left(\frac{\varepsilon_2}{16}\right)^{n}\varepsilon_1^{2l+1}\{2l(\xi+1)+\xi-1\},$ $G_s^{(q)}(\varepsilon_1,\varepsilon_2,\xi)=\varepsilon_2\sum_{n=\langle\frac{s}{2}\rangle}^{\infty}\sum_{l=\langle\frac{n}{2}\rangle}^{\infty}\frac{(-1)^{n+l-1}}{l}\binom{-\frac{1}{2}}{l-1}\binom{2l}{n}\binom{4n}{2n-s}\left(\frac{\varepsilon_2}{16}\right)^{n}\varepsilon_1^{2l+1}\{2l(\xi+1)+\xi-1\}\ \ \forall s\geq 1.$ (I – 3)
$\forall s\geq 0;0\langle\varepsilon_{1}\big(\langle 1;0\langle\xi\langle 1;q=1;\varepsilon_2=1$	$G_0^{(q)}(\varepsilon_1,\varepsilon_2,\xi)=2\varepsilon_1(1-\xi)+\varepsilon_2\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}\binom{-\frac{1}{2}}{l-1}\binom{4l}{2l}\left(\frac{\varepsilon_1}{4}\right)^{2l}\{2l(\xi+1)+\xi-1\},$ $G_s^{(q)}(\varepsilon_1,\varepsilon_2,\xi)=(-1)^{s}\varepsilon_1\sum_{l=\langle\frac{n}{2}\rangle}^{\infty}\frac{(-1)^{l}}{l}\left(\frac{\varepsilon_1}{4}\right)^{2l}\binom{4l}{2l-s}\binom{-\frac{1}{2}}{l-1}\{2l(\xi+1)+\xi-1\}\ \ \forall s\geq 1\ .$ (I – 4)

denotes the largest integer ≥ 1. [Note, although that, the only negative value of q in Equations (2.4) to (2.7) is –2, we considered the developments of the general case in which $q \notin z^+$]. Numerical results of the H's coefficients are listed in Table 2 for some values of q, ξ and $\varepsilon_{1,2}$. These results together with other experimentations show very rapid convergent Fourier series representation of $dg/d\theta_j^{(i)}$ in terms of the sectorial variables $\theta_j^{(i)}$ whatever the eccentricity ($\equiv\xi$) of the elliptic orbit may be.

TABLE 2. Values of $H_s^{(q)}(\varepsilon_1, \varepsilon_2, \xi)$; $s = 1(1)\,10$ for some values of q, ξ, $\varepsilon_{1,2}$.

$\varepsilon_1 = 0.8000$ $\varepsilon_2 = 0.2000$ $\xi = 0.9000$ $q = -2.0000$

s	H
1	0.44065860272E + 00
2	0.62158646779E – 01
3	0.87095936939E – 02
4	0.64476352879E – 03
5	0.72846741385E – 04
6	0.25660026556E – 05
7	0.51891471211E – 06
8	–0.15548915660E – 07
9	0.54042357971E – 08
10	–0.57576085479E – 09

TABLE 2. (continued)

$\varepsilon_1 = 0.3000$ $\varepsilon_2 = 1.0000$ $\xi = 0.3000$ $q = -0.4000$

s	H
1	0.69545295267E + 00
2	–0.28885735804E – 02
3	0.86258328202E – 03
4	–0.76190931124E – 04
5	0.10474024131E – 04
6	–0.32849644080E – 06
7	0.43801277008E – 07
8	–0.38660788371E – 08
9	0.34767083583E – 09
10	–0.22022276435E – 10

TABLE 2. (continued)

$\varepsilon_1 = 0.5000$ $\varepsilon_2 = 0.2000$ $\xi = 0.0100$ $q = 1.5000$

s	H
1	0.22212117726E + 00
2	–0.32926854543E – 02
3	0.16024031152E – 03
4	–0.54394097924E – 05
5	0.22779572851E – 06
6	–0.92014298699E – 08
7	0.38734360100E – 09
8	–0.16361872805E – 10
9	0.69997206128E – 12
10	–0.30122159870E – 13

TABLE 2. (continued)

$\varepsilon_1 = 0.3000$ $\varepsilon_2 = 0.8000$ $\xi = 0.4000$ $q = -6.0000$

s	H
1	030643843522E – 01
2	–0.54946288369E – 02
3	0.13214084682E – 02
4	–0.15600753053E – 03
5	0.20854576278E – 04
6	–0.18226477965E – 05
7	0.17275195878E – 06
8	–0.12052361102E – 07
9	0.91379948229E – 09
10	–0.56209221270E – 10

TABLE 2. (continued)

$\varepsilon_1 = 0.3000$ $\varepsilon_2 = 0.7000$ $\xi = 0.2000$ $q = 0.9000$

s	H
1	0.35733724803E + 00
2	-0.70672444442E – 02
3	0.10444687298E – 02
4	-0.27121910022E – 04
5	0.24064352426E – 05
6	–0.10635706309E – 06
7	0.77060812663E – 08
8	–0.41042100434E – 09
9	0.27396291890E – 10
10	–0.15962420706E – 11

TABLE 2. (continued)

$\varepsilon_1 = 0.0500$ $\varepsilon_2 = 0.0300$ $\xi = 0.9000$ $q = -0.3000$

s	H
1	0.59167089022E – 02
2	0.10212033767E – 05
3	–0.34376576208E – 08
4	–0.32077118542E – 11
5	0.41269026988E – 14
6	0.78754156951E – 17
7	–0.32695097357E – 20
8	–0.17117596518E – 22
9	0.00000000000E + 00
10	0.00000000000E + 00

TABLE 2. (continued)

$\varepsilon_1 = 0.8000$ $\varepsilon_2 = 0.0700$ $\xi = 0.9000$ $q = 2.7000$

s	H
1	0.27287282119E + 00
2	–0.31705040275E – 01
3	0.19239292663E – 02
4	–0.93033766220E – 04
5	0.45335953378E – 05
6	–0.23705642801E – 06
7	0.12982124095E – 07
8	–0.72873940980E – 09
9	0.41544321939E – 10
10	–0.23943022256E – 11

TABLE 2. (continued)

$\varepsilon_1 = 0.7000$ $\varepsilon_2 = 0.0700$ $\xi = 0.2000$ $q = -1.9000$

s	H
1	0.13747639819E + 00
2	–0.33513030170E – 03
3	0.41309866159E – 04
4	–0.14093014201E – 05
5	0.44428539143E – 07
6	–0.15521302413E – 08
7	0.54363282344E – 10
8	–0.19176943458E – 11
9	0.68293184065E – 13
10	–0.24472014023E – 14

2.2 Fourier Expansions of $g\grave{}$

Integrating Equations (2.3) we get

$$g\grave{} = \beta^{(i)}_{0,j,x} + \sum_{s=1}^{\infty} g^{(i)}_{s,j} \cos 2s\theta^{(i)}_j \; ; \; \forall (i-1).90^{\circ} \le \theta^{(i)}_j \le i.90^{\circ} \; ; \tag{2.9}$$

$$i = 1,2,3,4,; \quad j = 1,2,\ldots,N^{(i)} \; ,$$

where

$$g^{(i)}_{s,j} = -\,\nu\, \Lambda_{s,j,i} \,/\, 2s \; , \tag{2.10}$$

$\beta^{(i)}_{0,j,x}$ is the constant of integration during the xth revolution when the perturbed body in the jth sector that uses $\theta^{(i)}_j$ for its description. Before determining the constants of integration it is important to illustrate the following two types of the separating points between the elliptic sectors of the sectorial divisions of elliptic orbits.

2.2.1 The Types of Separating Points

(1) The first type of separating points (denoted symbolically as **sp1**) are those points on the elliptic orbit corresponding to the boundaries of elliptic sectors using the same representation equation (i.e., of the same i in $\theta^{(i)}$ for their descriptions. These **sp1** occur at $r^{(i)}_k$ between the elliptic sectors formed by $(r^{(i)}_{k-1}, r^{(i)}_k)$ and $(r^{(i)}_k, r^{(i)}_{k+1})$, $k = 2,3,\ldots, N^{(i)}$; $i = 1,2,3,4$. Clearly the **sp1** are the boundaries of elliptic sectors of unique sign. For the case in which $N^{(i)} = 1$ for any $i = 1,2,3$, or 4, the **sp1** does not exist for that value of i. The values of θ*\`s* at **sp1** points are

$$(\theta^{(i)}_{k-1} = i.90^{\circ} \; ; \; \theta^{(i)}_k = (i-1).90^{\circ}) \; ; \; i = 1,2,3,4 \; ; \; k = 2,3,\ldots, N^{(i)} \; . \tag{2.11}$$

(2) The second type of the separating points (denoted symbolically as **sp2**) are the four points on the elliptic orbit, each point corresponding to the common boundary between two consecutive elliptic sectors using different representation equations (i.e., of different i in $\theta^{(i)}$ for their descriptions. These **sp2** for any $N^{(i)} \ge 1$ occur at $r^{(2)}_1 = r^{(1)}_{N^{(1)}+1} = q$ (*say*) ; $r^{(3)}_1 = r^{(2)}_{N^{(2)}+1} = a(1+e)$; $r^{(4)}_1 = r^{(3)}_{N^{(3)}+1} = q'$ (*say*) and $r^{(1)}_1 = r^{(4)}_{N^{(4)}+1}\; a(1-e)$, which are the common boundaries of the elliptic sectors formed by $[(r^{(1)}_{N^{(1)}}, q), (q, r^{(2)}_2)]$; $[(r^{(2)}_{N^{(2)}}, a(1+e)), (a(1+e), r^{(3)}_2)]$,$[r^{(3)}_{N^{(3)}}, q\grave{}), (q\grave{}, r^{(4)}_2)]$ and $[r^{(4)}_{N(4)}, a(1-e)), (a(1-e), r^{(1)}_2)]$, respectively.

These pairs of sectors (each pair in square bracket) are respectively: two positive elliptic sectors; one positive and one negative elliptic sectors; two negative elliptic sectors; one negative and one positive elliptic sectors. The values of θ's at the four **sp2** points are

$$\begin{aligned}&(\theta_1^{(1)}=0^\circ\ ;\ \theta_{N^{(4)}}^{(4)}=360^\circ)\ ,\ (\theta_{N^{(1)}}^{(1)}=90^\circ\ ;\ \theta_1^{(2)}=90^\circ),\\&(\theta_{N^{(2)}}^{(2)}=180^\circ\ ;\ \theta_1^{(3)}=180^\circ)\ ,\ (\theta_{N^{(3)}}^{(3)}=270^\circ\ ;\ \theta_1^{(4)}=270^\circ)\ .\end{aligned}\quad (2.12)$$

2.2.2 Determination of The Constants $\beta_{0,j,x}^{(i)}$

In order to determine the constants $\beta_{0,j,x}^{(i)}$ the following two conditions are to be considered

(1) *The initial value of* g`. From the relation between $g$` and $\theta_j^{(i)}$ given by Equations (2.9) it is clear that, during a given revolution of the perturbed body the initial value could be considered as the value of g` at any arbitrary moment (corresponding to certain value of a sectorial variable). In this respect we shall adopt for this moment the perifocus passage of the perturbed body during the given revolution. Consequently, the initial value is the value of $g$` must have at $\theta_1^{(1)}=0^\circ$ (the beginning of the first sector in the direct orbital sense) during the xth revolution, let this value be denoted by c'_x. This means that, c' is a constant for each revolution of the perturbed body but varies from revolution to revolution.

(2) The second condition is the *continuity condition* at both types of separating points (**sp1** and **sp2** referred to the above) between the elliptic forming the the sectorial divisions situation.

Let us define the following constants

$$M_j^{(i)}=\sum_{s=1}^{\infty}g_{s,j}^{(i)}\ , \quad (2.13.1)$$

$$Q_j^{(i)}=-\sum_{s=1}^{\infty}(-1)^s g_{s,j}^{(i)} \quad (2.13.2)$$

By the first of the above conditions we get for the constant $\beta_{0,1,x}^{(1)}$ the formula

$$\beta_{0,1,x}^{(1)}=c`_x-M_{(1)}^{1}\ , \quad (2.14)$$

while by the second of the conditions (continuity condition) together with Equations (2.11) and (2.12) we get for the remaining constants the expressions

$$\beta_{0,n,x}^{(i)}=\beta_{0,n-1,x}^{(i)}+(-1)^i\{Q_{n-\delta_i}^{(i)}+M_{n+\delta_{i-1}}^{(i)}\}\ ;\ i=1,2,3,4\ ;\ n=2,3,\ldots,N^{(i)}\ , \quad (2.15)$$

$$\delta_i = \begin{cases} 1 & \text{if } i \text{ odd} \\ 0 & \text{if } i \text{ even} \end{cases} , \tag{2.16}$$

where the initial values for Equations (2.15) is given for $\beta^{(i)}_{0,1,x}$ by Equation (2.14), while for $i \geq 2$ these values are given by

$$\begin{aligned} \beta^{(2)}_{0,1,x} &= \beta^{(1)}_{0,N^{(1)},x} - Q^{(1)}_{N^{(1)}} + Q^{(2)}_1 , \\ \beta^{(3)}_{0,1,x} &= \beta^{(2)}_{0,N^{(2)},x} + M^{(2)}_{N^{(2)}} - M^{(3)}_1 , \\ \beta^{(4)}_{0,1,x} &= \beta^{(3)}_{0,N^{(3)},x} - Q^{(3)}_{N^{(3)}} + Q^{(4)}_1 . \end{aligned} \tag{2.17}$$

By successive use of Equations (2.15) we get for $\beta^{(i)}_{0,n,x}$ in terms of $c`_x$ the expressions

$$\beta^{(i)}_{0,n,x} = D^{(i)}_x + (-1)^i \left(\sum_{j=1}^{n-\delta_i} Q^{(i)}_j + \sum_{j=1}^{n} M^{(i)}_j \right) \; ; \; n = 2,3,\ldots, \; N^{(i)} \; , \tag{2.18.1}$$

$$D^{(i)}_x \begin{cases} = c`_x + \sum_{k=2}^{i} (-1)^{k-1} \sum_{j=1}^{N^{(k-1)}} (Q^{(k-1)}_j + M^{(k-1)}_j) \; ; \; \text{for } i = 2,3,4 \\ c`_x \; ; \; \text{for } i = 1. \end{cases} \tag{2.18.2}$$

By these equations we finally obtain for the constants of integration $\beta^{(i)}_{0,j,x}$ during the xth revolution the formula

$$\beta^{(i)}_{0,j,x} = c`_x + \frac{1}{2} g^{(i)}_{0,j} \; ; \; i = 1,2,3,4 \; ; \; j = 1,2,\ldots, N^{(i)} \; , \tag{2.19}$$

where $g^{(i)}_{0,j}$ are known in terms of M`s and $Q$`s constants.

It should be noted that $c`_x$ is related to $c`_0$ by

$$c`_x = c`_0 + 2\,\pi x\, v \, , \tag{2.20}$$

which enables us to express the constants of integration in terms of $c`_0$, which is the value of $g`$ must have at $\theta^{(1)}_1 = 0^\circ$ during the zero-th revolution.

From the above analysis, we finally deduce for $g`$ the Fourier expansions

$$\begin{aligned} g` &= c`_x + W^{(i)}_j \; , \\ W^{(i)}_j &= \frac{1}{2} g^{(i)}_{0,j} + \sum_{s=1}^{\infty} g^{(i)}_{s,j} \cos\theta^{(i)}_j \; ; \; i = 1,2,3,4 \; ; \; j = 1,2,\ldots, N^{(i)} , \end{aligned} \tag{2.21}$$

whatever the types and the number of sectors forming the divisions situation of the elliptic orbit may be.

2.2.3 Computational Algorithm

In what follows general broad lines of the computational algorithm for the coefficients $g_{s,j}^{(i)}\ \forall s \geq 0$ [of Equations (2.21)] are illustrated. For a given division situation of the elliptic orbit the computational steps are

1 – compute the η\`*s* and $l$\`*s* parameters using Equations (1.7) to (1.10).

2 – By the aid of Table 1, compute for the possible values of η\`*s* and $l$\`*s* the G\`s functions, then $H$\`s functions from Equation (2.8) and hence the Λ\`*s* coefficients from Equations (2.4) to (2.7).

3 – compute $g_{s,j}^{(1)}\ \ \forall s \geq 1\ ;\ i = 1,2,3,4\ ;\ j = 1,2,..,N^{(i)}$ from Equation (2.10).

4 – compute $M_j^{(i)}$ and $Q_j^{(i)}$ from Equations (2.13).

5 – compute $g_{0,j}^{(i)}$ using Equations (2.18).

3. Expansions of cos mg\` and sin mg\`

In this section, cos mg\` and sin mg\` are completely developed from the analytical and the computational points of views.

3.1 Analytical Developments

In what follows, the literal analytical expressions for the trigonometric series representations of cos mg\` and sin mg\` where *m* is positive integer will be developed in terms of $\theta_j^{(i)}$ and $c_x^{`}$. The materials of the present subsection will be established in the following theorem:

Theorem 5. The trigonometric series representations of cos mg\` and sin mg\` in terms of $\theta_j^{(i)}$ and $c_x^{`}$ for all (i–1). 90º ≤ $\theta_j^{(i)} \leq i.$ 90º ; i = 1,2,3,4 ; j = 1,2,..., $N^{(i)}$ and $x \geq 0$ are

$$\cos mg^{`} = \cos mc_x^{`} \left\{ \frac{1}{2} F_{0,j,m}^{(i)} + \sum_{s=1}^{\infty} F_{s,j,m}^{(i)} \cos 2s\theta_j^{(i)} \right\} - \sin mc_x^{`} \times \times \left\{ \frac{1}{2} P_{0,j,m}^{(i)} + \sum_{s=1}^{\infty} P_{s,j,m}^{(i)} \cos 2s\theta_j^{(i)} \right\}, \tag{3.1}$$

$$\boxed{\begin{aligned}&\sin\ m\grave{g} = \cos\ m\grave{c}_x\ \left\{\frac{1}{2}P^{(i)}_{0,j,m} + \sum_{s=1}^{\infty} P^{(i)}_{s,j,m}\cos 2s\theta^{(i)}_j\right\} + \sin\ m\grave{c}_x\ \times\\ &\times\left\{\frac{1}{2}F^{(i)}_{0,j,m} + \sum_{s=1}^{\infty} F^{(i)}_{s,j,m}\ \cos 2s\theta^{(i)}_j\right\}\ ,\end{aligned}} \tag{3.2}$$

where m is positive integer,

$$F^{(i)}_{0,j,m} = 2 + \sum_{r=1}^{\infty}\frac{(-1)^r\ m^{2r}}{(2r)!}A^{(i)}_{0,j,2r}\ , \tag{3.3.1}$$

$$F^{(i)}_{s,j,m} = \sum_{r=1}^{\infty}\frac{(-1)^r\ m^{2r}}{(2r)!}\ A^{(i)}_{s,j,2r}\quad \forall s \geq 1\ , \tag{3.3.2}$$

$$P^{(i)}_{s,j,m} = \sum_{r=1}^{\infty}\frac{(-1)^r\ m^{2r-1}}{(2r-1)!}A^{(i)}_{s,j,2r-1}\quad \forall s \geq 0\ , \tag{3.3.3}$$

$$A^{(i)}_{s,j,1} \equiv g^{(i)}_{s,j}\quad \forall s \geq 0\ , \tag{3.4.1}$$

$$\begin{aligned}A^{(i)}_{s,j,r} = 2\sum_{p=s}^{\infty}2^{-2p}\binom{2p}{p-s}\prod_{k=1}^{r-1}\left[\sum_{p_k=0}^{p_{k-1}}2^{2\delta_k}\left\{\frac{1}{2}g^{(i)}_{0,j} + (-1)^{\delta_k}\sum_{n=1+\delta_k}^{\infty}(-1)^n g^{(i)}_{n,j}\left(\frac{n}{n-\delta_k}\right)\binom{n-1+\delta_k}{2\delta_k}\right\}\right]\times\\ \times\left[2^{2p_{r-1}}\left\{\frac{1}{2}g^{(i)}_{0,j} + (-1)^{p_{r-1}}\sum_{n=1+p_{r-1}}^{\infty}(-1)^n g^{(i)}_{n,j}\left(\frac{n}{n-p_{r-1}}\right)\binom{n-1+p_{r-1}}{2p_{r-1}}\right\}\right],\ \forall r>1,\ s\geq 0\ ,\end{aligned} \tag{3.4.2}$$

$\delta_k = p_{k-1} - p_k$, $p_0 = p,$ and g\`s coefficients are those defined by Equation (2.21).

Proof. Recalling Equation (2.21) as

$$W^{(i)}_j = \frac{1}{2}A^{(i)}_{0,j,1} + \sum_{s=1}^{\infty}A^{(i)}_{s,j,1}\ \cos 2s\theta^{(i)}_j\ , \tag{3.5}$$

where

$$A^{(i)}_{s,j,1} \equiv g^{(i)}_{s,j}\ . \tag{3.6}$$

Let us first convert Equation (3.5) into power polynomial in $\cos\theta^{(i)}_j$, and this in accordance with Section 4 of Paper V could be written as

$$W_j^{(i)} = \sum_{s=0}^{\infty} B_{s,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s} \ , \qquad (3.7)$$

where

$$B_{s,j,1}^{(i)} = 2^{2s}\left\{\frac{1}{2}A_{0,j,1}^{(i)} + (-1)^s \sum_{l=s+1}^{\infty} (-1)^l A_{l,j,1}^{(i)} \left(\frac{l}{l-s}\right)\binom{l+s-1}{2s}\right\} \ . \qquad (3.8)$$

Second, let us find the expansion of $[W_j^{(i)}]^r$ in powers of $\cos\theta_j^{(i)}$, r being a positive integer. Now

$$[W_j^{(i)}]^2 = \sum_{s_1=0}^{\infty} B_{s_1,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s_1} \ . \ \sum_{s=0}^{\infty} B_{s,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s} \ .$$

Let $s + s_1 = s`$ then

$$[W_j^{(i)}]^2 = \sum_{s_1=0}^{\infty} B_{s_1,j,1}^{(i)} \ \cdot \ \sum_{s=s_1}^{\infty} B_{s-s_1,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s}$$

$$= \sum_{s=0}^{\infty} [\cos\theta_j^{(i)}]^{2s} \ . \ \sum_{s_1=0}^{\infty} B_{s_1,j,1}^{(i)} B_{s-s_1,j,1}^{(i)} \ ,$$

that is

$$[W_j^{(i)}]^2 = \sum_{s=0}^{\infty} B_{s,j,2}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \ , \qquad (3.9)$$

where

$$B_{s,j,2}^{(i)} = \sum_{s_1=0}^{s} B_{s-s_1,j,1}^{(i)} B_{s_1,j,1}^{(i)} \ . \qquad (310)$$

Next

$$[W_j^{(i)}]^3 = \sum_{s_1=0}^{\infty} B_{s_1,j,2}^{(i)} \ [\cos\theta_j^{(i)}]^{2s_1} \ . \ \sum_{s=0}^{\infty} B_{s,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s}$$

$$= \sum_{s_1=0}^{\infty} B_{s_1,j,2}^{(i)} \ \cdot \ \sum_{s=s_1}^{\infty} B_{s-s_1,j,1}^{(i)} [\cos\theta_j^{(i)}]^{2s}$$

$$= \sum_{s=0}^{\infty} [\cos_j^{(i)}]^{2s} \ \cdot \ \sum_{s_1=0}^{s} B_{s-s_1,j,1}^{(i)} \ B_{s_1,j,2}^{(i)} \ ,$$

that is

$$[W_j^{(i)}]^3 = \sum_{s=0}^{\infty} B_{s,j,3}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \ , \tag{3.11}$$

where

$$B_{s,j,3}^{(i)} = \sum_{s_1=0}^{s} B_{s-s_1,j,1}^{(i)} \ B_{s_1,j,2}^{(i)} \ . \tag{3.12}$$

We now assume for r positive integer

$$[W_j^{(i)}]^r = \sum_{s=0}^{\infty} B_{s,j,r}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \ , \tag{3.13}$$

where

$$B_{s,j,r}^{(i)} = \sum_{s_1=0}^{s} B_{s-s_1,j,1}^{(i)} \ B_{s_1,j,r-1}^{(i)} \ . \tag{3.14}$$

and shall show that this form holds also for the expansion of $[W_j^{(i)}]^{r+1}$.

From Equation (3.14) we have

$$\begin{aligned} [W_j^{(i)}]^{r+1} &= \sum_{s_1=0}^{\infty} B_{s_1,j,r}^{(i)} \ [\cos\theta_j^{(i)}]^{2s_1} \ . \ \sum_{s=0}^{\infty} B_{s,j,1}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \\ &= \sum_{s_1=0}^{\infty} B_{s_1,j,r}^{(i)} \ . \ \sum_{s=s_1}^{\infty} B_{s-s_1,j,1}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \\ &= \sum_{s=0}^{\infty} [\cos\theta_j^{(i)}]^{2s} \ . \ \sum_{s=s_1}^{s} B_{s-s_1,j,1}^{(i)} \ B_{s_1,j,r}^{(i)} \ , \end{aligned}$$

that is

$$[W_j^{(i)}]^{r+1} = \sum_{s=0}^{\infty} B_{s,j,r+1}^{(i)} \ [\cos\theta_j^{(i)}]^{2s} \ , \tag{3.15}$$

where

$$B_{s,j,r+1}^{(i)} = \sum_{s_1=0}^{s} B_{s-s_1,j,1}^{(i)} \ B_{s_1,j,r}^{(i)} \ , \tag{3.16}$$

which shows that $B_{s,j,r+1}^{(i)}$ is of the same form as $B_{s,j,r}^{(i)}$.

We shall now express $B_{s,j,r}^{(i)}$ in terms of $B_{s,j,1}^{(i)}$. Using Equation (3.14) as a recurring formula, we have

$$B^{(i)}_{s,j,r} = \sum_{s_1=0}^{s} B^{(i)}_{s-s_1,j,1} \sum_{s_2=0}^{s_1} B^{(i)}_{s_1-s_2,j,1} B^{(i)}_{s_1,j,r-2} \tag{3.17}$$

$$= \sum_{s_1=0}^{s} B^{(i)}_{s-s_1,j,1} \sum_{s_2=0}^{s_1} B^{(i)}_{s_1-s_2,j,1} \sum_{s_3=0}^{s_2} B^{(i)}_{s_2-s_3,j,1} B^{(i)}_{s_3,j,r-3}$$

$$= \sum_{s_1=0}^{s} B^{(i)}_{s-s_1,j,1} \sum_{s_2=0}^{s_1} B^{(i)}_{s_1-s_2,j,1} \cdots \sum_{s_{r-1}=0}^{s_{r-2}} B^{(i)}_{s_{r-2}-s_{r-1},j,1} B^{(i)}_{s_{r-1},j,1} \ ,$$

finally we have for the coefficients $B^{(i)}_{s,j,r}$ [of Equation (3.13)] in terms of $B^{(i)}_{s,j,1}$ the expression

$$B^{(i)}_{s,j,r} = \prod_{k=1}^{r-1} \left(\sum_{s_k=0}^{s_{k-1}} B^{(i)}_{s_{k-1}-s_k,j,1} \right) B^{(i)}_{s_{r-1},j,1} \ , r > 1, \ s_0 = s \ . \tag{3.18}$$

Now, by using the identity

$$(\cos\psi)^{2s} = 2^{-2s} \left\{ \binom{2s}{s} + 2\sum_{l=0}^{s-1} \binom{2s}{l} \cos(2s-2l)\psi \right\} ,$$

Equation (3.13) reduces to

$$[W^{(i)}_j]^r = \frac{1}{2} A^{(i)}_{0,j,r} + \sum_{s=1}^{\infty} A^{(i)}_{s,j,r} \ \cos 2s\theta^{(i)}_j \ , \tag{3.19}$$

where

$$A^{(i)}_{s,j,r} = 2\sum_{l=s}^{\infty} 2^{-2l} \binom{2l}{l-s} B^{(i)}_{l,j,r} \ \ \forall \ \ s \geq 0 \ , \ \ r > 1 \ . \tag{3.20}$$

Using Equations (3.8) and (3.18) into Equation (3.20), Equation (3.4.2) follows directly.

From Equation (2.21) we have

$$\cos \text{mg}` = \cos \text{m}c_x` \cos \text{m}w^{(i)}_j - \sin \text{m}c_x` \sin \text{m}w^{(i)}_j . \tag{3.21.1}$$

$$\sin \text{mg}` = \cos \text{m}c_x` \sin \text{m}w^{(i)}_j + \sin \text{m}c_x` \cos \text{m}w^{(i)}_j \tag{3.21.2}$$

Using Equation (3.19) into the Taylor's series, expansions of cos $mw^{(i)}_j$ and sin $mw^{(i)}_j$ we get

$$\cos \ mw^{(i)}_j = \frac{1}{2} F^{(i)}_{0,j,m} \ . + \ \sum_{s=1}^{\infty} F^{(i)}_{s,j,m} \ \cos 2s\theta^{(i)}_j \ , \tag{3.22.1}$$

$$\sin\ m w_j^{(i)} = \frac{1}{2} P_{0,j,m}^{(i)}\ .\ +\ \sum_{s=1}^{\infty} F_{s,j,m}^{(i)}\ \cos 2s\theta_j^{(i)}\ , \qquad (3.22.2)$$

where *F`s* and *P`s* coefficients are those given by Equations (3.3)

Finally, by Equations (3.22) into Equations (3.21), Equations (3.1) and (3.2) follows immediately. Q.E.D.

3.2 Computational Developments

In this subsection, some recurrence formulae will be established to facilitate the computations of the coefficients included in the above analytical formulations.

3.2.1 Recurrence Formulae for A`S Coefficients

For $r > 1$, Equation (3.19) could be written as

$$[W_j^{(i)}]^r\ =\ [W_j^{(i)}]\ .\ [W_j^{(i)}]^{r-1}\ ,$$

that is

$$\frac{1}{2} A_{0,j,r}^{(i)} + \sum_{s=1}^{\infty} A_{s,j,r}^{(i)}\ \cos 2s\theta_j^{(i)} = \left\{ \frac{1}{2} A_{0,j,1}^{(i)}\ + \sum_{s=1}^{\infty} {}_{s,j,1}^{(i)}\ \cos 2s\theta_j^{(i)} \right\}\ .\ \left\{ \frac{1}{2} A_{0,j,r-1}^{(i)} + \right.$$

$$\left. + \sum_{s=1}^{\infty} A_{s,j,r,-1}^{(i)}\ \cos 2s\theta_j^{(i)} \right\}\ ,$$

from which we deduce the recurrence formulae

$$A_{s,j,r}^{(i)} = \frac{1}{2} A_{0,j,1}^{(i)}\ .\ A_{s,j,r-1}^{(i)}\ +\ \frac{1}{2} \sum_{k=1}^{\infty} A_{k,j,1}^{(i)}\ .\left\{ A_{k+s,j,r-1}^{(i)} + A_{k+s,j,r-1}^{(i)} \right\}\ . \qquad (3.23)$$

This equation is to be used with the following conditions

$$A_{l,j,m}^{(i)} = \begin{cases} 2 & \text{if } \ l = 0,\ \ m = 0, \\ 0 & \text{if } \ l \neq 0,\ m = 0, \\ A_{-l,j,m}^{(i)} & \forall l > 0. \end{cases} \qquad (3.24)$$

3.2.2 Recurrence Formulae for F`s and P`s Coefficients

Let $R_{s,j,m}$ stands for $F_{s,j,m}$ or $P_{s,j,m}$, then by the aid of the identities

$$\cos k\Psi = 2\cos(k-1)\ \Psi \cos \Psi - \cos(k-2)\ \Psi,$$

$$\sin k\Psi = 2\sin(k-1)\ \Psi \cos \Psi - \sin(k-2)\ \Psi$$

and Equations (3.22) we deduce the recurrence formulae

$$R^{(i)}_{s,j,m} = F^{(i)}_{0,j,1}\ .\ R^{(i)}_{s,j,m-1} + \sum_{l=1}^{\infty} F^{(i)}_{l,j,m}\left\{R^{(i)}_{s+l,j,m-1} + R^{(i)}_{s-l,j,m-1}\right\} - R^{(i)}_{s,j,m-2}\ . \quad (3.25)$$

In order to apply these formulae we have to take into account the following conditions

$$F^{(i)}_{k,j,m} = \begin{cases} 2 & \text{if } k=0,\ m=0 \\ 0 & \text{if } k\neq 0,\ m=0 \\ F^{(i)}_{-k,j,m} & \forall\ k>0. \\ F^{(i)}_{k,j,-m} & \forall\ k \end{cases} \quad (3.26)$$

$$P^{(i)}_{k,j,m} = \begin{cases} 0 & \text{if } m=0,\ \forall\ k, \\ P^{(i)}_{-k,j,m} & \forall\ k>0, \\ -P^{(i)}_{k,j,-m} & \forall\ k \end{cases} \quad (3.27)$$

3.2.3 Recurrence Formulae for B`s Coefficients

Recalling Equations (3.7) and (3.13) as

$$\Phi(z) \equiv W^{(i)}_{j} = \sum_{s=0}^{\infty} B^{(i)}_{s,j,1}\ z^{s}\ , \quad (3.28)$$

$$F(z) \equiv [w^{(i)}_{j}]^{r} = \sum_{s=0}^{\infty} B^{(i)}_{s,j,r}\ z^{s}\ , \quad (3.29)$$

where $z \equiv [\cos\theta^{(i)}_{j}]^{2}$, and r positive integer.

From these equations with $z = 0$ we get

$$B^{(i)}_{0,j,r} = [B^{(i)}_{0,j,1}]^{r}\ . \quad (3.30)$$

From Equation (3.29) we have

$$r \ln \Phi(z) = \ln F(z)\ ,$$

then on differentiating this equation with respect to z we get

$$r\Phi`(z)\, F(z) = \Phi(z)\, F`(z)\ . \quad (3.31)$$

Differentiating Equation (3.31) k times with respect to z and then applying Leibnitz's rule for the kth derivative of a product we deduce that

$$r\sum_{l=0}^{k}\binom{k}{l}\Phi^{(l+1)}(z)F^{(k-l)}(z)=\sum_{l=0}^{k}\binom{k}{l}F^{(l+1)}(z)\,\Phi^{(k-l)}(z)\ , \qquad (3.32)$$

where for example

$$\Phi^{(l)}(z)\equiv\frac{d^l\Phi(z)}{dz^l}\ .$$

Equation (3.32) is valid at any z, consequently at $z = 0$ we deduce by using Equations (3.28), (3.29) and

$$\binom{k}{l}\ (l+1)!\ (k-l)!\ =\ (l+1)\ \ k!$$

that

$$r\sum_{l=0}^{k}(l+1)\ \ B^{(i)}_{l+1,j,1}B^{(i)}_{k-l,j,r}=\sum_{l=0}^{k}(l+1)\ \ B^{(i)}_{k-l,j,1}\ \ B^{(i)}_{l+1,j,r}\ . \qquad (3.33)$$

At this stage it is required to relate $B^{(i)}_{k+1,j,r.}$ in terms of $B^{(i)}_{q,j,r}$, $q \leq k$, and this could be performed by writing Equation (3.33) as

$$(k+1)\ B^{(i)}_{0,j,1}\ \ B^{(i)}_{k+1,j,r}=r\sum_{l=0}^{k}(l+1)\ \ B^{(i)}_{l+1,j,1}\ \ B^{(i)}_{k-l,j,r}-\sum_{l=0}^{k-1}(l+1)\ \ B^{(i)}_{k-l,j,1}\ \ B^{(i)}_{l+1,j,r}\ . \qquad (3.34)$$

From this equation we have

$$kB^{(i)}_{0,j,1}\ \ B^{(i)}_{k,j,r}=r\sum_{l=1}^{k}l\ \ B^{(i)}_{l,j,1}\ \ B^{(i)}_{k-l,j,r}-\sum_{l=1}^{k-1}(k-l)\ \ B^{(i)}_{l,j,1}\ \ B^{(i)}_{k-l,j,r}\ ,$$

then by adding the zero value $\{-(k-k)\ B^{(i)}_{k,j,1}\ \ B^{(i)}_{0,j,r}\}$ to the right hand side of the above equation, we deduce for the B`s coefficients the recurrence formulae

$$\begin{aligned}B^{(i)}_{0,j,r}&=[B^{(i)}_{0,j,1}]^r\ ,\\ B^{(i)}_{k,j,r}&=\frac{1}{kB^{(i)}_{0,j,r}}\ \sum_{l=1}^{k}(rl-k+l)\ \ B^{(i)}_{l,j,1}\ \ B^{(i)}_{k-l,j,r}\quad\forall k\geq 1\ .\end{aligned} \qquad (3.35)$$

Conclusions

In conclusion, in the present paper the literal analytical expressions of the mean anomaly g` of the disturbing body are established in terms of $\theta^{(i)}_j$ and $c^`_x$.

Also, cos mg` and sin mg` were completely developed from the analytical and the computational points of views. For the analytical developments, the exact literal expressions of the coefficients for the doubly trigonometric series representations of these functions were established while for the computational developments, recurrence formulae were also established to facilitate digital computations. All the formulations developed in the present paper are general in the sense that they are valid whatever the types and the number of sectors forming the divisions of the elliptic orbit may be. In addition they are also valid during any revolution of the perturbed body in its Keplerian orbit.

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النظرية المفكوكية للحركة الإهليلجية ذات الاختلاف المركزي ونصف المحور الأكبر الاختياريين XIII تحسين الحصة المتوسطة g´ للجسم المقلق و sin mg´ & cos mg´ في النظرية النظامية للقطاعات

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المملكة العربية السعودية

المستخلص. في هذا البحث من بحوث السلسلة بدأنا الخطوة الرابعة من النظرية النظامية للقطاعات وذلك بتشييد الصيغ التحليلية الحرفية والصيغ المثلثية الثنائية ، وعلاوة على ذلك فقد شيدت بعض الصيغ التعاوديَّة لتسهيل الحسابات العددية لمعاملات التمثيل التسلسلي ، والجدير بالذكر أن جميع الصيغ المستحدثة في هذا البحث تعتبر عامة لكونها تتحقق مهما كان نوع وعدد القطاعات التي تكوِّن التقسيمات للمدار الإهليلجي . بالإضافة إلى ذلك فإنها تصلح خلال أي دورة للجسم المقلق في مداره الكبلري . وأخيراً فقد اشتمل البحث أيضًا على بعض النتائج العددية للمعاملات .