Space Dynamics with Mathematica:
I. Initial Value Problem for Parabolic Motion

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ABSTRACT. In this paper of the series, the initial value problem for parabolic motion is considered analytically and computationally. A Mathematica program for this problem is established together with the powerful modified top-down continued fraction evaluation algorithm for the ratio of two hypergeometric functions. Numerical applications of the program are also included.

1. Introduction

The main goal of this series of papers is to use Mathematica to solve typical problems in space dynamics. The usages will be through the graphics capability, and the symbolic and numerical computing features of Mathematica.

In the present paper of the series, we consider the initial value problem for parabolic motion throughout the solution of the generalized Barker's equation by using continued fraction evaluation.

In general, the initial value problem of space dynamics may be formulated as: Given position and velocity vectors r_s and v_n at the initial time t_n, and given a second time t_l, find r_l and v_l. On the other hand, the formulations in terms of

Cartesian coordinates add an important feature from a practical standpoint. Because, the observations are lengths and velocities directly expressed in Cartesian coordinates.

Many instances of parabolic orbits occur in the solar system, especially some comets[1]. Also, the intermediate portion of interplanetary mission may be a heliocentric parabola[2]. The initial value problem of parabolic motion will be given in Subsection 2.3.

For the parabolic motion, the basic equation to be solved is known as Barker's equation. There are many methods to solve this equation[3]; one of the most powerful method is that uses continued fraction evaluation. The usages of the continued fraction expansions in modern computations are increasing, because, they are generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series[4]. Barker's equation in its most conveniently generalized form and its continued fraction solution are considered in Subsections 2.1 and 2.2.

There are several methods available for the evaluation of continued fractions. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n'th convergent were accumulated separately with three-term recurrence formulae. The drawback to the first method is, obviously, having to decide far down the fraction to begin in order to ensure convergence. The drawback to the second method is that the numerator and denominator rapidly overflow or underflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm which works from top down while avoiding numerical difficulties would be ideal from a programming standpoint. Gautschi's[5] top-down algorithm with Shepperd's[6] modification is considered in Subsection 3.1.

A computational algorithm of the present problem is given with its numerical applications in Section 3. Finally, a Mathematica program of the algorithm is given in Appendix I.

2. Formulations

2.1 Generalized Barker's Equation

The relation between the true anomaly f and the time t for parabolic motion is known as Barker's equation[3] and is given as

$$\tan^3 \frac{1}{2} f + 3 \tan \frac{1}{2} f = 2H, \tag{2.1}$$

where

$$H = 3\sqrt{\frac{\mu}{p^3}}\,(t-\tau), \tag{2.2}$$

μ the gravitational constant, τ the time of pericenter passage, and p is twice the pericenter distance q.

For the two epochs t_l and t_n (1 > n), Equation (2.1) is most conveniently expressed in terms of the variable

$$\chi = \sigma_l - \sigma_n, \tag{2.3}$$

as

$$6\sqrt{\mu}\,(t_l - t_n) = \chi^3 + 3\sigma_n\chi^2 + 6r_n\chi, \tag{2.4}$$

where for the position and velocity vectors r_s and v_s at the epoch $t = t_s$ we have

$$\sigma_s = \frac{1}{\sqrt{\mu}} <r_s, v_s> = \sqrt{p}\tan\frac{1}{2}f_s, \tag{2.5}$$

f_s is the value of f at $t = t_s$, and $< A, B >$ is used to denote the scalar product of two vectors A and B. The magnitudes of the position vectors r_l and r_n at the epochs t_l and t_n are related by

$$r_l = r_n + \sigma_n\chi + \frac{1}{2}\chi^2. \tag{2.6}$$

Equation (2.4) is the generalized form of Barker's equation. Finally, since the polar equation of a parabolic orbit is

$$r = \frac{p}{2}(1 + \tan^2\frac{1}{2}f),$$

according to equation (2.5) and the constancy of p we get

$$p = 2r_l - \sigma_l^2 = 2r_n - \sigma_n^2. \tag{2.7}$$

It can be shown that, the solution of the generalized form of Barker's Equation (2.4) is

$$\chi = \sqrt{p}\tan\frac{1}{2}f - \sigma_n, \tag{2.8}$$

where $\tan\frac{1}{2}f$ is the solution of

$$\tan^3\frac{1}{2}f + 3\tan\frac{1}{2}f = 2B \tag{2.9}$$

and

$$B = p^{-\frac{3}{2}}[\sigma_n(r_n + p) + 3\sqrt{\mu}\,(t_1 - t_n)]. \tag{2.10}$$

Equation (2.9) is exactly of the same form as the classical Barker's Equation (2.1), so that all solution methods developed for any of them are applicable to the other one.

2.2 Continued Fraction Solution

The solution of the cubic Equation (2.9) can be written as[4],

$$\tan \frac{1}{2} f = \frac{2B}{3} G, \qquad (2.11)$$

with

$$G = \frac{F(\frac{2}{3}, \frac{4}{3}, \frac{3}{2}; -B^2)}{F(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}; -B^2)} \qquad (2.12)$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function[8].

It is well-known[4] that the Gauss continued fraction may be written in the following form

$$U(a, b, c, z) = \frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} \qquad (2.13)$$

$$= \frac{1}{1-} \frac{h_1 z}{1-} \frac{h_2 z}{1-} \cdots \qquad (2.14)$$

where

$$h_{2n} = \frac{(n+b)\ (n+c-a)}{(2n+c-1)\ (2n+c)} \qquad (2.15)$$

and

$$h_{2n+1} = \frac{(n+a)\ (n+c-b)}{(2n+c)\ (2n+c+1)} \ . \qquad (2.16)$$

Comparing Equations (2.12) and (2.13) we have

$$G = U\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, -B^2\right), \qquad (2.17)$$

which together with Equation (2.11) represent the continued fraction solution of Equation (2.9).

2.3 Initial Value Problem for Parabolic Motion

Let (r_n, v_n) and (r_l, v_l) be the position and velocity vectors at $t = t_n$ and $t = t_l$ respectively. Then the transition matrix $\Phi_{l,\,n}$, needed to extrapolate r_l, v_l from r_n, v_n is given by

$$\Phi_{l,n} = \begin{bmatrix} F_{l,n} & G_{l,n} \\ F_{t,l,n} & G_{t,l,n} \end{bmatrix},$$

where its elements are known as Lagrangian coefficients. Although that these coefficients are of different forms for the different types of conic motion, the transition matrix for all types satisfies the same two properties

1 – Its determinant is equal to one.

2 – For any three points on an orbit r_k, r_m, r_s

$$\Phi_{sk} = \Phi_{sm} \; \Phi_{mk}.$$ (2.18)

The Lagrangian coefficients for parabolic motion are given[3] by

$$F_{l,n} = 1 - \frac{\chi}{2r_n},$$ (2.19)

$$G_{l,n} = \frac{\chi}{2\sqrt{\mu}}(2r_n + \sigma_n\chi),$$ (2.20)

$$F_{t,l,n} = -\frac{\sqrt{\mu}\,\chi}{r_l\,r_n},$$ (2.21)

$$G_{t,l,n} = 1 - \frac{\chi^2}{2r_l}.$$ (2.22)

3. Computational Developments

3.1. Top-Down Continued Fraction Evaluation

Gautschi[5] proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as

$$c = \frac{n_1}{d_1 +} \frac{n_2}{d_2 +} \frac{n_3}{d_3 +} \dots$$

then initialize the following parameters

$$a_1 = 1$$

$$b_1 = \frac{n_1}{d_1}$$

$$c_1 = \frac{n_1}{d_1}$$

and iterate ($k = 1,2,$) according to

$$a_{k+1} = \frac{1}{1 + [\frac{n_{k+1}}{d_k d_{k+1}}]a_k}$$

$$b_{k+1} = [a_{k+1} - 1]\; b_k,$$

$$c_{k+1} = c_k + b_{k+1}.$$

In the limit, the *c* sequence converges to the value of the continued fraction. However, the complexity of the coefficients given by Equations (2.21) and (2.22) make it somewhat inefficient to use Gautschi's algorithm directly. Sheppered[6], instead, with the introduction of a couple of additional parameters, Gautschi's method may be reformatted into a form more suitable for digital computers.

The method proceeds as follows, first initialize

$$k = 1 - 2(a - b),$$

$$l = 2(c - 1),$$

$$d = 4c(c - 1),$$

$$n = 4b(c - a),$$

$$A = 1,$$

$$B = 1,$$

$$U = 1,$$

then evaluate these parameters recursively using

$$k \leftarrow - k,$$

$$l \leftarrow l + 2,$$

$$d \leftarrow d + 4l,$$

$$n \leftarrow n + (1 + k)l,$$

$$A \leftarrow \frac{d}{(d - nAz)},$$

$$B \leftarrow (A - 1)B,$$

$$U \leftarrow U + B,$$

3.2 Computational Algorithm

- **Purpose:** To solve the initial value problem for parabolic motion.
- **Input:** $r_n(x_n, y_n, z_n)$, $v_n(\dot{x}_n, \dot{y}_n, \dot{z}_n)$, μ, t_n, t_l.
- **Output:** $r_l(x_l, y_l, z_l)$, $v_l(\dot{x}_l, \dot{y}_l, \dot{z}_l)$.

● **Computational Sequence**:

1 – $r_n = \sqrt{x_n^2 + y_n^2 + z_n^2}$.

2 – $\sigma_n = \dfrac{1}{\sqrt{\mu}}(x_n \dot{x}_n + y_n \dot{y}_n + z_n \dot{z}_n)$.

3 – $p = 2r_n - \sigma_n^2$

4 – B from Equation (2.10).

5 – Solve Baker' s Equation (2.9) for $\tan\dfrac{1}{2}f$ by the

  continued fraction algorithm of Section 3.1

6 – x from Equation (2.8).

7 – r_l from Equation (2.6).

8 – $F_{l,n}, G_{l,n}, F_{t,l,n}$ and $G_{t,l,n}$ from Equations (2.19) to (2.22).

9 – $r_l = F_{l,n}\, r_n + G_{l,n}\, v_n$.

10 – $v_1 = F_{t,l,n}\; r_n + G_{t,l,n}\; v_n$.

11 – The algorithm is completed.

The Mathematical coding IVPPM of the above algorithm is listed in Appendix I.

3.3 Numerical Applications

The purpose of this subsection is to illustrate the numerical computing features of Mathematica program IVPPM to solve the problem of the present paper.

3.3.1. Test orbits

We consider six orbits all with the eccentricity $e = 1$ (parabolic orbits) and $\mu = 1$, while the other orbital elements are listed for each orbit in the first column of Tables 1 to 6, where q, e, τ have their meaning as mentioned

previously, $i \equiv$ orbital inclination, $\Omega \equiv$ longitude of the ascending node, $w \equiv$ argument of the pericenter. The corresponding initial position and velocity vectors r_n and v_n at $t = t_n$ are given in the second columns of the tables. These vectors are obtained by the classical transformations[3] from the orbital elements to r and v vectors for parabolic orbits.

TABLE 1. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 1.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1185$	Check
$q = 2$	$x = + 08.472708887852480$	$x = + 06.487780921114990$	$|\Delta q| = 0$
$e = 1$	$y = - 03.889171171426470$	$y = - 03.99518874359456$	$|\Delta e| = 0$
$\tau = 1200$	$z = - 04.891720757166960$	$z = - 03.74572206124905$	$|\Delta \tau| = 0$
$i = 150°$	$\dot{x} = - 00.376464633037897$	$\dot{x} = - 00.420267180735975$	$|\Delta i| = 0$
$\Omega = 270°$	$\dot{y} = - 00.031620986067439$	$\dot{y} = - 00.008169836661225$	$|\Delta \Omega| = 0$
$w = 60°$	$\dot{z} = + 00.217351957224803$	$\dot{z} = + 00.24264136992948$	$|\Delta w| = 1.98952 * 10^{-13}$

TABLE 2. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 2.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1190$	Check
$q = 3$	$x = + 06.322692305028100$	$x = + 03.79921539196239$	$|\Delta w| = 1.02141 * 10^{-14}$
$e = 1$	$y = - 06.744186230117650$	$y = - 02.42769413144328$	$|\Delta e| = 0$
$\tau = 1200$	$z = - 03.856510613020800$	$z = - 04.08274796329054$	$|\Delta \tau| = 0$
$i = 64.28°$	$\dot{x} = - 00.209010213428531$	$\dot{x} = - 00.312683041576236$	$|\Delta i| = 0$
$\Omega = 121.56°$	$\dot{y} = - 00.391070377884770$	$\dot{y} = - 00.47960235539925$	$|\Delta \Omega| = 0$
$w = 318.94°$	$\dot{z} = - 00.055193843742839$	$\dot{z} = + 00.03200193755009$	$|\Delta w| = 0$

TABLE 3. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 3.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1195$	Check
$q = 4$	$x = + 05.032890490439500$	$x = - 03.65646695547321$	$|\Delta q| = 0$
$e = 1$	$y = - 02.022899299973510$	$y = - 04.02593638772547$	$|\Delta q| = 0$
$\tau = 1210$	$z = - 11.92238686576550$	$z = - 05.82601657179870$	$|\Delta \tau| = 0$
$i = 71.71°$	$\dot{x} = - 00.067174384911609$	$\dot{x} = - 00.128146409099325$	$|\Delta i| = 0$
$\Omega = 248.49°$	$\dot{y} = - 00.149367259977560$	$\dot{y} = - 00.10166283237762$	$|\Delta \Omega| = 0$
$w = 39.44°$	$\dot{z} = + 00.354779670796995$	$\dot{z} = + 00.4734820117564$	$|\Delta w| = 9.9476 * 10^{-14}$

TABLE 4. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 4.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1200$	Check
$q = 5$	$x = -12.635055870473000$	$x = -05.07285913130296$	$|\Delta q| = 0$
$e = 1$	$y = +02.865497787435040$	$y = -01.01697056570985$	$|\Delta e| = 0$
$\tau = 1215$	$z = -06.099886579216900$	$z = -06.22318246544142$	$|\Delta \tau| = 0$
$i = 62.85°$	$\dot{x} = +00.320597480273014$	$\dot{x} = +00.453789981558589$	$|\Delta i| = 0$
$\Omega = 333.25°$	$\dot{y} = -00.186915433326709$	$\dot{y} = -00.19348546708922$	$|\Delta \Omega| = 0$
$w = 316.16°$	$\dot{z} = -00.044091080915244$	$\dot{z} = +00.061369005164287$	$|\Delta w| = 0$

TABLE 5. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 5.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1225$	Check
$q = 6$	$x = -13.3462851499139$	$x = +05.015759601461220$	$|\Delta q| = 0$
$e = 1$	$y = +02.40121603068125$	$y = -03.48881849117271$	$|\Delta e| = 0$
$\tau = 1220$	$z = -07.53256517109402$	$z = -01.67279606461874$	$|\Delta \tau| = 0$
$i = 62.85°$	$\dot{x} = +00.31230839096072$	$x. = +00.354437575700268$	$|\Delta i| = 0$
$\Omega = 333.25°$	$\dot{y} = -00.174630187259768$	$y. = +00.0686189170411564$	$|\Delta \Omega| = 0$
$w = 316.16°$	$\dot{z} = -00.02997113230710$	$z. = +00.430567853146194$	$|\Delta w| = 0$

TABLE 6. (r_n, v_n) and (r_l, v_l) with their check for the Test orbit no. 6.

Orbital elements	$r_n, v_n ; t_n = 1180$	$r_l, v_l ; t_l = 1230$	Check
$q = 7$	$x = -15.730098191062700$	$x = +01.94170524933400$	$|\Delta q| = 0$
$e = 1$	$y = +03.275550095550990$	$y = -06.67810022702522$	$|\Delta e| = 2.9976*10^{-15}$
$\tau = 1225$	$z = +04.481143778551210$	$z = -02.04523909038297$	$|\Delta \tau| = 0$
$i = 18.5°$	$\dot{x} = +00.193392621818427$	$\dot{x} = +00.509846367177894$	$|\Delta i| = 0$
$\Omega = 44.7°$	$\dot{y} = -00.265845093885484$	$\dot{y} = +00.073902601440295$	$|\Delta \Omega| = 0$
$w = 221.4°$	$\dot{z} = -00.108741437343389$	$\dot{z} = -00.102417386748446$	$|\Delta w| = 0$

3.3.2. Numerical Results

The IVPPM program was applied to the six test orbits for the position and velocity vectors r_l and v_l at $t = t_l$. The results are listed for each orbit in the third column of Tables 1 to 6. To check the accuracy of the computed vectors,

we make use of the fact that, in the Keplerian motion the position and velocity vectors at any instant can produce the constant elements of the orbital motion. In this respect, we utilize r_l and v_l in the basic Equations[3] for the conversion to the orbital elements, and the absolute difference of each of the resulting element and the corresponding initial element (the first columns of the Tables) is listed in the fourth columns of Tables 1 to 6.

Finally, it should be mentioned that, if the time difference $T = t_l - t_n$ is sufficiently large, we might have to repeat the process by decrementing T several times. The transition matrices thus sequentially generated are, of course, multiplied together [according to Equation (2.18)] to produce the final desired matrix.

In concluding the present paper, an accurate Mathematica program for the initial value problem for parabolic motion is established together with the powerful modified top-down continued fraction evaluation algorithm for the ratio of two hypergeometric functions.

References

[1] **Marsden, B.G.,** *Minor Planet Circular No. 16001* (1990).

[2] **Escobal, P.R.,** *Methods of Astrodynamics,* Wiley, New York (1968).

[3] **Battin, R.H.,** *An Introduction to the Mathematics and Methods of Astrodynamics,* AIAA Education Series (1987).

[4] **Wall, H.S.,** *Analytic Theory of Continued Fraction,* D. van Nostrand Co., New York (1948).

[5] **Gautschi, W.,** Computational Aspects of Three-Term Recurrence Relations, *SIAM Review,* **9,** No. 1 (1967).

[6] **Shepperd, S.W.,** Universal Keplerian State Transition Matrix, *Celestial Mechanics,* **35,** 129 (1985).

[7] **Danby, J.M.A.,** *Fundamentals of Celestial Mechanics,* Willmann Bell, Inc., Richmond, Virginia, U.S.A. (1988).

[8] **Abramowitz, Moaned Stein, IA,** *Handbook of Mathematical Functions,* Dover Publications, Inc., New York (1972).

Appendix I

Mathematica Program IVPPM

%EVALUATION OF B GIVEN BY EQUATION (2.10)

%Notation : rn $\rightarrow$ rn , xn $\rightarrow$ xn , xnd $\rightarrow \dot{x}_n$ etc.

rn = Sqrt [xn^2+yn^2+zn^2] ;

sn = (1/ Sqrt [mu])*(xn*xnd+yn*ynd+zn*znd) ;

p = 2*rn-sn^2 ;

BB = N[p^(-3/2)*(sn*(rn+p)+3*Sqrt[mu]*(tl-tn)),15]

% CONTINUED FRACTION SOLUTION

tol = 10^(-14) ;

AL = 2/3 ;

BE = 1/3 ;

GA = 1/2

X = -B^2 ;

AK = 1-2*(AL-BE) ;

BL = 2*(GA-1) ;

AD = 4*GA*(GA-1) ;

AN = 4*BE*(GA-AL) ;

A = 1;

BB = 1;

G = 1;

(Label[20]; AK=-AK;BL=BL+2;AD=AD+4*BL;\

AN = AN+(1+AK)*BL; A=AD/(AD-AN*A*X);\

BB = (A-1)*BB; G=G+BB; If [Abs [BB]>tol,Go to[20]])

Y1 = (2/3)*BB*G

%POSITION AND VELOCITY VECTORS AT TIME t = tl

x = Sqrt [p]*Y1-sn

r1 = rn+sn*X+X^2/2

F = 1-(X^2/(2*rn))

G = X/(2*Sqrt [mu])*(2*rn+sn*X)

Ft = -Sqrt [mu]/(r1*rn)*X

Gt = 1-(X^2/(2*r1))

x1 = F*xn+G*xnd

y1 = F*yn+G*ynd

z1= F*zn+G*znd

x1d = Ft*xn+Gt*xnd

y1d = Ft*yn+Gt*ynd

z1d = Ft*zn+Gt*znd

الديناميكا الفضائية باستخدام (MATHEMATICA)
١ – مسألة القيمة الابتدائية للحركة المكافئـة

محمد عادل عبد العزيز شرف و عبد النبي سعد و عمرو عبد العزيز شرف
و سميحة عبد الله عباس نجم الدين القاري

قسم العلوم الفلكية ، جامعة الملك عبد العزيز ، جدة – المملكة العربية السعودية ، قسم الفلك ، المعهد القومي للبحوث الفلكية والفيزيائية ، القاهرة ، مصر ، قسم الهندسة الرياضية والفيزيائية ، كلية الهندسة جامعة القاهرة ، مصر و قسم العلوم والرياضيات ، كلية إعداد المعلمات ، مكة المكرمة – المملكة العربية السعودية

المستخلص. في هذا البحث من سلسلة البحوث شُيِّدت مسألة القيمة الابتدائية للحركة في القطع المكافئ تحليليًا وحسابيًا .

وقد استخدم (MATHEMATICA) لهذه المسألة مع طريقة (السفلي – العلوي) المعدلة لحساب الكسر المتصل للنسبة بين دالتين فوق هندسيتين ، واحتوت أيضًا على تطبيقات عددية للبرنامج .