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# Matrix Transformation into a New Sequence Space Related to Invariant Means

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*Abstract.* In this paper we define a sequence space $V_\infty$ through the concept of invariant means and prove that this is a Banach space under certain norm. We further characterize the matrix classes ($l_\infty$, $V_\infty$) and ($l_1$, $V_\infty$).

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## Introduction and Preliminaries

Let $l_\infty$ and $c$ be the Banach spaces of bounded and convergent sequences $x = (x_k)$ respectively with norm $\|x\|_\infty = \sup_{k\geq 0} |x_k|$, and $l_1$ be the space of absolutely convergent series with $\|x\|_1 = \sum_k |x_k|$.

Let $\sigma$ be a mapping of the set of positive integers $\mathbb{N}$ into itself. A continuous linear functional $\phi$ on $l_\infty$ is said to be an *invariant mean* or a $\sigma$*-mean* if and only if, (i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$, (ii) $\phi(e) = 1$, where $e = (1, 1, 1, \cdots)$, and (iii) $\phi(Tx) = \phi(x)$ for all $x \in l_\infty$, where $Tx = (Tx_k) = (x_{\sigma(k)})$. In case $\sigma$ is the translation mapping $k \to k + 1$, a $\sigma$-mean is often called a Banach limit[1] and $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set $f$ of almost convergent sequences[2].

Note that[3],

$$V_\sigma := \{ x \in l_\infty : \lim_m t_{mn}(x) = L \quad \text{uniformly in } n, L = \sigma\text{- } \lim x \},$$

where

$$t_{mn}(x) = (x_n + Tx_n + \cdots + T^m x_n) / (m+1),$$

and

$$t_{-1,n} = 0.$$

A $\sigma$-mean extends the limit functional on $c$ in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if $\sigma$ has no finite orbits, that is to say, if and only if, for all $n \geq 0$, $m \geq 1$, $\sigma^m(n) \neq n$ (see Ref. [4]).

We say that a bounded sequence $x$ is *$\sigma$-convergent* if and only if $x \in V_\sigma$ such that $\sigma^m(n) \neq n$ for all $n \geq 0$, $m \geq 1$ (see Ref. [5]).

Let $X$ and $Y$ be two sequence spaces and $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of real complex numbers. We write $Ax = (A_n(x))$ where $A_n(x) = \sum_k a_{nk}x_k$ and the series converges for each $n$. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that $A$ defines a matrix transformation from $X$ into $Y$. By $(X, Y)$ we denote the class of matrices $A$ such that $Ax \in Y$ for $x \in X$.

In this paper we define a new sequence space $V_\infty$ related to the concept of $\sigma$-mean and prove that $V_\infty$ in a Banach space under certain norm. We also characterize the matrices of the class $(l_\infty, V_\infty)$ and $(l_1, V_\infty)$.

We define the space $V_\infty$ as follows

$$V_\infty := \{ x \in l_\infty : \sup_{m,n} |t_{mn}(x)| < \infty \}.$$

Note that if $\sigma$ is a translation then $V_\infty$ is reduced to the space

$$f_\infty := \{ x \in l_\infty : \sup_{m,n} |g_{mn}(x)| < \infty \}.$$

where

$$g_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^{\infty} x_{k+n}.$$

We call the space $V_\infty$ as the space of *$\sigma$-bounded sequences*. It is clear that $c \subset V_\sigma \subset V_\infty \subset l_\infty$.

## Results

***Theorem 1***

$V_\infty$ is a Banach space normed by

$$\|x\| = \sup_{m,n} |t_{mn}(x)| \tag{1}$$

***Proof***

It is easy to see that $V_\infty$ is a normed linear space under the norm in (1).

Now we have to show the completeness of $V_\infty$. Let $(x^{(i)})_{i=1}^{\infty}$ be a Cauchy sequence in $V_\infty$. Then $(x_k^{(i)})_{i=1}^{\infty}$ is Cauchy sequence in $\mathbb{R}$ for each $k$ and hence convergent in $\mathbb{R}$ that is, $x_x^{(i)} \to x_k$, say, as $i \to \infty$. Let $x = (x_k)_{k=1}^{\infty}$. Then by the definition of norm on $V_\infty$, we can easily show that

$$\|x^{(i)} - x\| \to 0 \ as \ i \to \infty .$$

Now, we have to show that $x \in V_\infty$. Since $(x^{(i)}$ is a Cauchy sequence, given $\varepsilon > 0$, there is a positive integer $N$ depending upon $\varepsilon$ such that, for each $i$, $r > N$,

$$\|x^{(i)} - x^{(r)}\| \ \varepsilon .$$

Hence

$$\sup_{m,n} | t_{mn}(x^{(i)} - x^{(r)}) | < \varepsilon.$$

This implies that

$$|t_{mn}(x^{(i)} - x^{(r)}| < \varepsilon, \tag{2}$$

for each $m$, $n$; or

$$|L^{(i)} - L^{(r)}| < \varepsilon \tag{3}$$

for each $i$, $r > N$ ; where $L^{(i)} = \sigma - \lim x^{(i)}$. Let $L = \lim_{r\to\infty} L^{(r)}$. Then the $\sigma$-mean of $x$, $\phi(x) = \lim_i \phi(x^{(i)}) = \lim_i L^{(i)} = L$. Letting $r \to \infty$ in (2) and (3), we get

$$|t_{mn}(x^{(i)} - x)| \leq \varepsilon, \text{ for each } m, n; \tag{4}$$

and

$$|L^{(i)} - L| \leq \varepsilon , \tag{5}$$

for $i > N$. Now, fix $i$ in the above inequalities. Since $x^{(i)} \in V_\infty$ for fixed $i$, we obtain

$$\lim_m t_{mn}(x^{(i)}) = L^{(i)}, \quad \text{uniformly in } n.$$

Hence, for a given $\varepsilon$, there exists a positive integer $m_0$ (depending upon $i$ and $\varepsilon$ but not on $n$) such that

$$|t_{mn}(x^{(i)} - L^{(i)}| < \varepsilon, \tag{6}$$

for $m \geq m_0$ for all $n$. Now, by (4), (5) and (6), we get

$$|t_{mn}(x)-L| \le |t_{mn}(x)-t_{mn}(x^{(i)})|+|t_{mn}(x^{(i)})-L^{(i)}|+|L^{(i)}-L|< 3\varepsilon,$$

for $m \ge m_0$ and for all $n$. Hence $x \in V_{\sigma}$. Since $V_\sigma \subset V_\infty$, $x \in V_\infty$. This completes the proof of the theorem.

Let $Ax$ be defined. Then, for all $m$, $n \ge 0$, we write

$$t_{mn}(Ax)=\sum_{k=1}^{\infty} t(n,k,m)x_k ,$$

where,

$$t(n,k,m)=\frac{1}{m+1}\sum_{j=0}^{\infty} a(\sigma^j(n),k),$$

and $a(n, k)$ denotes the element $a_{nk}$ of the matrix $A$.

***Theorem 2***

$A \in (l_\infty, V_\infty)$ if and only if

$$\sup_{m,n}\sum_k |t(n,k,m)|<\infty. \tag{7}$$

***Proof***

Sufficiency. Let (7) hold and $x \in l_\infty$. Then we have

$$\begin{aligned}|t_{mn}(Ax)| &\le \sum_k |t(n,k,m)x_k| \\ &\le (\sum_k |t(n,k,m)|)(\sup_k |x_k|).\end{aligned}$$

Now, taking the supremum over $m$, $n$ on both sides, we get $Ax \in V_\infty$ for $x \in l_\infty$, *i.e.*, $A \in (l_\infty, V_\infty)$.

Necessity. Let $A \in (l_\infty, V_\infty)$. Write $q_n(x) = \sup_m |t_{mn}(Ax)|$. It is easy to see that for $n \ge 0$, $q_n$ is a continuous seminorm on $l_\infty$ and $(q_n)$ is pointwise bounded on $l_\infty$. Suppose (7) is not true. Then there exists $x \in l_\infty$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities[5], the set

$$\{x \in l_\infty : \sup_n q_n(x)=\infty\}$$

is of second category in $l_\infty$ and hence nonempty, that is, there is $x \in l_\infty$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that $(q_n)$ is pointwise bounded on $l_\infty$. Now, by the Banach-Steinhauss theorem, there is a constant $M$ such that

$$q_n(x) \le M\,\|x\|_1. \tag{8}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \operatorname{sgn}\ t(n,k,m) & \text{for each } n,m \text{ and } 1 \le k \le k_0, \\ 0 & \text{for } k > k_0. \end{cases}$$

Then $x \in l_\infty$. Applying this sequence to (8), we get (7).

This completes the proof of the theorem.

If $\sigma$ is a translation, then by the above theorem, we obtain

***Corollary 3***

$A \in (l_\infty, f_\infty)$ if and only if

$$\sup_{m,n} \sum_k \frac{1}{m+1} \left| \sum_{j=0}^{m} a_{n+j,k} \right| < \infty.$$

***Theorem 4***

$A \in (l_1, V_\infty)$ if and only if

$$\sup_{n,k,m} |t(n,k,m)| < \infty. \tag{9}$$

***Proof***

Sufficiency. Suppose that $x = (x_k) \in l_1$. We have

$$|t_{mn}(Ax)| \le \sum_k |t(n,k,m)x_k|$$

$$\le (\sup_k |t(n,k,m)|) \ (\sum_k |x_k|).$$

Taking the supremum over $n$, $m$ on both sides and using (9), we get $Ax \in V_\infty$ for $x \in l_1$.

Necessity. Let us define a continuous linear functional $Q_{mn}$ on $l_1$ by

$$Q_{mn}(x) = \sum_k t(n,k,m)x_k.$$

Now,

$$|Q_{mn}(x)| \le \sup_k |t(n,k,m)| \, \|x\|_1 .$$

and hence

$$\|Q_{m,n}\| \le \sup_k |t(n,k,m)|. \tag{10}$$

For any fixed $k \in \mathbb{N}$, define $x = (x_i)$ by

$$x_i = \begin{cases} \operatorname{sgn}\ t(n,k,m) & \text{for } i = k, \\ 0 & \text{for } i \ne k. \end{cases}$$

Then $\|x\|_1 = 1$, and

$$|Q_{mn}(x)| = |t(n,k,m)x_k| \\ = |t(n,k,m)| \;\; \|x\|_1,$$

hence

$$\|Q_{mn}(x)\| \geq \sup_k |t(n,k,m)|. \tag{11}$$

By (10) and (11), we get

$$\|Q_{mn}(x)\| = \sup_k |t(n,k,m).$$

Since $A \in (l_1, V_\infty)$, we have, for $x \in l_1$,

$$\sup_{m,n} |Q_{m,n}(x)| = \sup_{m,n} |\sum_k t(n,k,m)x_k| < \infty.$$

Hence, by the uniform boundedness principle, we have

$$\sup_{m,n} \|Q_{m,n}(x)\| = \sup_{m,n,k} |t(n,k,m)| < \infty.$$

This complete the proof of the theorem.

If we take $\sigma(n) = n + 1$ in the above theorem, we get

***Corollary 5***

$A \in (l_1, f_\infty)$ if and only if

$$\sup_{n,k,m} \frac{1}{m+1} |\sum_{j=0}^{m} a_{n+j,k}| < \infty.$$

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# محول المصفوفات إلى فراغ متسلسلات جديد

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*المستخلص.* في هذا البحث تم تعريف الفراغ $\Lambda_\infty$ من خلال مفهوم (Invariant Means) ويثبت أن هـــذا الفـراغ هــو من فـراغـات بانــاخ (Banach Space). أيضًا نقوم بتصنيف الفراغات $(\Lambda_\infty , l_\infty)$ و $(\Lambda_\infty , l_1)$.