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On Pairwise Semi-Generalized Closed Sets

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Abstract. Fukutake introduced and investigated the notion of generalized closed sets in bitopological spaces. We also use these concepts to introduce the new notions of some operator as well as ij-generalized semi-closure, ij-semi-generalized closure, ij-generalized semi-interior and ij-semi-generalized interior. As continuation of the study of generalized closed sets in bitopological spaces, in this paper, we introduce and study the class of semi-generalized closed sets which are properly placed between the classes of generalized semi-closed sets and semi-closed sets. We shall consider a fundamental property of pairwise semi-generalized closed sets. Applying pairwise semi-generalized closed set, we investigate the notion of pairwise semi $T_{1/2}$-space. Also, we introduce ij-semi-generalized continuous maps and ij-semi-generalized irresolute maps.

Keywords: ij-semi-open set, ij-semi-generalized closed set, ij-semi closure, ij-semi $T_{1/2}$-space, ij-semi-generalized function.

Introduction

For the first time the concepts of generalized closed sets and $T_{1/2}$-spaces were defined by Levine[1] in 1970. In 1987, Bhattacharyya and Lahiri[2] introduced a new class of sets called semi-generalized closed sets by replacing the closure operator in the original Levine's definition by

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2000 AMS Mathematics Subject classification: 54A10, 54D10, 54E55 semi-closure operator and replacing openness of the superset with semi-openness. It was observed that the notion of generalized closed and semi-generalized closed sets are independent to each other. In 1986, Fukutake[3] introduced and studied generalized closed sets in bitopological spaces. Also, he defined a new closure operator and strongly pairwise $T_{1/2}$-spaces.

In 1990, Arya and Nour[4] defined generalized semi-closed sets. Generalized closed and generalized semi-closed sets are independent notions. The notion of generalized α-closed sets was introduced recently by Maki, *et al.* [5] .

The aim of this paper is to continue the study of the above mentioned classes of sets by introducing the notion of semi-generalized closed sets in bitopological spaces. Also, we study the basic properties of this concept. The relations between this concept and the other classes of generalized closed sets will be investigated. As applications of ij-semi-generalized closed sets, we introduce and study some notions like ij-generalized semi-closure (semi-interior) and ij-semi-generalized closure (interior) operators and ij-semi $T_{1/2}$-spaces. We introduce as well the notion of generalized semi-continuous function and study the relation between the newly defined concepts with related ones.

Throughout this paper, (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, ν_1, ν_2) (or briefly X, Y and Z) denote bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, we shall denote the closure of A and the interior of A with respect to τ_i (or σ_i) by i-cl(A) and i-int(A) respectively for i=1,2. Also i,j=1,2 and $i \neq j$.

A subset A of a bitopological space X is said to be ij-semi-open[6], if there exists a τ_i-open set U of X such that $U \subset A \subset$ j-cl(U), or equivalently if $A \subset$ j-cl(i-int(A)). The complement of an ij-semi-open set is said to be ij-semi-closed. The family of all ij-semi-open sets of X is denoted by ij-SO(X) and for $x \in X$, the family of all ij-semi-open sets containing x is denoted by ij-SO(X,x). An ij-semi-interior[6] of A, denoted by ij-sint(A), is the union of all ij-semi-open sets contained in A. The intersection of all ij-semi-closed sets containing A is called the ij-semi-closure[6] of A and denoted by ij-scl(A). A subset A of X is said to be ij-α-open[7] if $A \subset$ i-int(j-cl(i-int(A))).

Now, we mention the following definitions and results:

Definition 1.1. A subset A of a space X is called an ij-generalized closed[3] (briefly ij-g-closed) if j-cl(A) $\subset$ U, whenever $A \subset U$ and U is τ_i -open in X.

Lemma 1.2. For every subset A of a space X, we have the following:

(i) X \ ij-sint(A) = ij-scl(X\A).

(ii) X \ ij-scl(A) = ij-sint(X\A).

(ii) ij-sint(A) $\cap$ ij-sint(B) = ij-sint(A $\cap$ B).

Proof. (i) Let $x \notin$ ij-scl(X\A), Then there exists U$\in$ij-SO(X) containing x such that $U \cap (X\backslash A) = \phi$. Thus $x \in U \subset A$ and $x \in$ ij-sint(A). Hence $x \notin$X\ij-sint(A). Now, let $x \notin$X\ij-sint(A). Thus $x \in$ ij-sint(A) and there exists U$\in$ij-SO(X) such that $x \in U \subset A$. Hence $U \cap (X\backslash A) = \phi$ and $x \notin$ij-scl(X\A).

(ii) The proof is similar to that of (i).

(iii) Since $A \subset B$, then ij-sint(A) $\subset$ ij-sint(B). So, $A \cap B \subset A$ implies that ij-sint(A $\cap$ B) $\subset$ ij-sint(A) and $A \cap B \subset B$ implies ij-sint(A $\cap$ B) $\subset$ ij-sint(B). Thus ij-sint(A $\cap$ B) $\subset$ ij-sint(A) $\cap$ ij-sint(B). Now, let $x \in$ij-sint(A) $\cap$ ij-sint(B). Thus $x \in$ij-sint(A) and $x \in$ ij-sint(B). Then $x \in A$ and $x \in B$. Hence $x \in$ij-sint(A$\cap$B) and ij-sint(A) $\cap$ ij-sint(B) $\subset$ ij-sint(A $\cap$ B).

Definition 1.3. A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called

(i) ij-semi-continuous[6] if $f^{-1}(V)$ is an ij-semi-open set of X for every σ_i-open set V of Y, equivalently $f^{-1}(V)$ is an ij-semi-closed set of X for every σ_i-closed set V of Y.

(ii) ij-irresolute[8] if $f^{-1}(V) \in$ ij-SO(X) for every V $\in$ ij-SO(Y).

(iii) ij-pre-semi-open (resp. ij-pre-semi-closed) if f (U) is ij-semi-open in Y for every ij-semi-open set U in X (resp. if f (U) is ij-semi-closed for every ij-semi-closed set U in X).

Lemma 1.4[8]**.** A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is ij-irresolute if and only if for every subset A of X, f(ij-scl(A)) $\subset$ ij-scl(f(A)).

Lemma 1.5. A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is ji-irresolute, then for every subset B of Y, ji-scl(f^{-1}(B)) $\subset$ f^{-1}(ji-scl(B)).

Proof. Let x$\in$ji-scl(f^{-1}(B)). Suppose that V is ji-semi-open set of Y containing f(x), *i.e.* f(x)$\in$V, then x$\in$ f^{-1}(V). Since f^{-1}(V) is ji-semi-open of X, then $f^{-1}(V) \cap f^{-1}(B) \neq \phi$ implies.

that $f^{-1}(V\cap B) \neq \phi$ and $V\cap B \neq \phi$. Thus f(x)$\in$ ji-scl(B) and x$\in$ f^{-1}(f(x)) $\in$ f^{-1}(ji-scl(B)), this means that x$\in$ f^{-1}(ji-scl(B)). Hence ji-scl(f^{-1}(B)) $\subset$ f^{-1}(ji-scl(B)).

Lemma 1.6. A bijection f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is ij-pre-semi-open if and only if f is ij-pre-semi-closed.

Proof. Let F be an ij-semi-closed set of X. Then F = X\U, where U is an ij-semi-open set. Hence f(F) = f(X\U) = Y\f(U). Since f is ij-pre-semi-open, then f(U) is ij-semi-open and Y\f(U) is ij-semi-closed in Y. Thus f(F) is ij-semi-closed and f is ij-pre-semi-closed. The proof of the converse is similar.

Lemma 1.7. If a function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-closed, then for each subset S $\subset$ Y and each τ_i-open set U containing f^{-1}(S), there is a σ_i -open set V containing S such that $f^{-1}(V) \subset U$.

Proof. Let S $\subset$ Y and U is τ_i-open containing f^{-1}(S). Put V = Y \ f(X\U). Then V is σ_i -open set in Y containing S. It follows from a straightforward calculation that $f^{-1}(V) \subset U$.

Lemma 1.8. If a function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is ij-pre-semi-closed, then for each subset S $\subset$ Y and each U$\in$ij-SO(X) containing f^{-1}(S), there exists V$\in$ ij-SO(Y) such that S $\subset$ V and $f^{-1}(V) \subset U$.

Proof. Let S $\subset$ Y and U$\in$ij-SO(X) containing f^{-1}(S). Put V = Y \ f(X\U). Then V is ij-semi-open set in Y containing S. It follows from a straightforward calculation that $f^{-1}(V) \subset U$.

2. Basic Properties of ij-Semi-Generalized Closed Sets

Definition 2.1. A subset A of a space X is called ij-semi-generalized closed (briefly ij-sg-closed) if ji-scl(A) $\subset$ U whenever A $\subset$ U and U $\in$ ij-

SO(X,x). If $A \subset X$ is 12-sg-closed and 21-sg-closed, then it said to be pairwise semi-generalized-closed (briefly P-sg-closed).

The complement of ij-semi-generalized closed set is called ij-semi-generalized open (briefly ij-sg-open). The collection of all ij-sg-closed (resp. ij-sg-open) subsets of a given space (X, τ_1, τ_2) is denoted by SGC(X)(resp. SGO(X)).

Definition 2.2. A subset A of a space X is called an ij-generalized α-closed set (briefly, ij-gα-closed) if ji-αcl(A)$\subset$ U whenever A$\subset$ U and U is ij-α-open in X. If $A \subset X$ is 12-gα-closed and 21-gα-closed, then it is said to be pairwise generalized α-closed (briefly P-gα-closed).

Remark 2.3. Every τ_j-closed set is ij-g-closed but the converse is not true, the following example shows that.

Example 2.4. Let X = {a, b, c}; τ_1= { ϕ, {a}, X} and τ_2 { ϕ, {a}, {a, b}, X}. Then A = {a, b} is 12-g-closed, but it is not τ_2-closed since τ_2-cl(A) = X.

Remark 2.5. The notions of ij-g-closed sets and ij-sg-closed sets are independent to each other. To see this, we have the following examples.

Example 2.6. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and τ_2={ ϕ, {a,b}, X}. If A = {b}, then A is 12-g-closed set but it is not 12-sg-closed set since 21-scl(A) = X.

Example 2.7. Let $\tau_1 = \tau_2$ be the usual topology on the real line R and let A be the open interval (a, b). Then A is 12-sg-closed but not 12-g-closed.

Theorem 2.8. Every ji-semi-closed set is ij-sg-closed.

Proof. A set A$\subset$ X is ji-semi-closed set if and only if ji-scl(A) = A. Thus ji-scl(A) $\subset$ U for every U $\in$ ij-SO(X,x) and $A \subset U$.

The following example shows that the converse of Theorem 2.8 is not true.

Example 2.9. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a, d\}, \{a, b, d\}, X\}$ and $\tau_2 = \{\phi, \{a, b, c\}, X\}$. If A = {a, b, c}, then 21-scl(A) = X and so, A is not 21-semi-closed but A is 12-sg-closed.

Remark 2.10. The union of two ij-sg-closed sets need not be ij-sg-closed, the following example shows that.

Example 2.11. Let X = {a, b, c, d}, τ_1= {ϕ, {a}, {b},{a, b}, X} and τ_2 = {ϕ, {a, c}, {b, d}, X}. If A = {a}, B = {b}, then 21-scl(A) = {a} and 21-scl(B) = {b}. So A, B are 21-semi-closed and 12-sg-closed. But $A \cup B$ is not 12-sg-closed since 21-scl({a, b}) = X $\not\subset$ {a, b} $\in$ 12-SO(X).

Recall that a subset A of a space X is called ij-clopen set[9] if it both i-closed and j-open.

Remark 2.12. The product of two ij-sg-closed sets need not be ij-sg-closed, even in the case when one of ij-sg-closed sets is ji-clopen. The following example shows that.

Example 2.13. Let X = {a, b, c}, τ_1 = { ϕ, {a, b}, X} and τ_2 = {ϕ, {c}, X}. Set A = {b, c}. A is 12-sg-closed of a space (X, τ_1, τ_2). But A×X is not 12-sg-closed of a space (X×X, $\tau_1\times\tau_1$, $\tau_2\times\tau_2$). Set U = X×X\ {(a, c)} and A×X $\subset$ U, where U is 12-semi-open in X×X but 21-scl(A×X) = X×X $\not\subset$ U.

Remark 2.14. Every ji-semi-open set is ij-sg-open but the following example shows that the converse is not true.

Example 2.15. Let X = {a, b, c, d}, τ_1= {ϕ, {a, d}, {a, b, d}, X} and τ_2 = {ϕ,{a, b, c}, X}. If A = {a, b, c}, then 21-scl(A) = X and so A is not 21-semi-closed but A is 12-sg-closed. Then X\A = {d} is 12-sg-open but not 21-semi-open.

Theorem 2.16. If A is ij-sg-closed and A $\subset$ B $\subset$ ji-scl(A), then B is ij-sg-closed.

Proof. Let B $\subset$ U, where U is ij-semi-open. Since A is ij-sg-closed and A $\subset$ B it follows that A $\subset$ U. By hypothesis B $\subset$ ji-scl(A) and hence ji-scl(B) $\subset$ ji-scl(A) $\subset$ U. Thus B is ij-sg-closed.

Theorem 2.17. In a space X, ij-SO(X) = ji-SC(X) if and only if every subset of X is pairwise sg-closed.

Proof. Let A$\subset$ X such that A $\subset$ U where U$\in$ij-SO(X). Then U$\in$ji-SC(X) and therefore, ji-scl(A) $\subset$ ji-scl(U) = U which shows that A is ij-sg-closed.

Conversely, let $U \in$ ij-SO(X). Since by hypothesis every subset of X is ij-sg-closed, U is ij-sg-closed. This shows that ji-scl(U) $\subset$ U and so U = ji-scl(U). This implies $U \in$ ji-SC(X). On the other hand, let $F \in$ ji-SC(X). Then $X \backslash F \in$ ji-SO(X). Since $X \backslash F$ is ji-sg-closed, then ij-scl($X \backslash F$) $\subset$ $X \backslash F$. Hence $X \backslash F \in$ ij-SC(X) and $F \in$ ij-SO(X).

Lemma 2.18. For each point x of (X, τ_1, τ_2), {x} is ij-semi-closed or $X \backslash \{x\}$ is ij-sg-closed.

Proof. Let x be a point of X. Suppose that {x} is not ij-semi-closed. Since $X \backslash \{x\}$ is not ij-semi-open. Then the only ij-semi-open containing $X \backslash \{x\}$ is X. Thus we have ji-scl ($X \backslash \{x\}$) $\subset$ X and $X \backslash \{x\}$ is ij-sg-closed.

Lemma 2.19. If A is ij-sg-closed, then ji-scl(A)\A contains no non-empty ij-semi-closed set.

Proof. Let F be an ij-semi-closed set such that F $\subset$ ji-scl(A)\A. Then F $\subset$ X\A and so A $\subset$ X\F. Since A is ij-sg-closed, ji-scl(A) $\subset$ X\F and so F $\subset$ X\ji-scl(A). Thus F $\subset$ (X\ji-scl (A)) $\cap$ ji-scl (A) = ϕ. As a result, F is empty.

Corollary 2.20. Let A be ij-sg-closed. Then A is ji-semi-closed if and only if ji-scl(A)\A is ij-semi-closed.

Proof. Let A be ij-sg-closed. If A is ji-semi-closed. Then ji-scl(A)\A = ϕ which is ij-semi-closed.

Conversely, let ji-scl(A)\A be ij-semi-closed and A be ij-sg-closed. Then ji-scl(A)\A does not contain any non empty ij-semi-closed subset, since ji-scl(A)\A is ij-semi-closed, ji-scl(A)\A = ϕ which implies that A is ji-semi-closed.

Theorem 2.21. A subset A of a bitopological space (X, τ_1, τ_2) is ij-sg-open if and only if F $\subset$ ji-sint(A) whenever F is ij-semi-closed and F $\subset$ A.

Proof. Let A be ij-sg-open and suppose F $\subset$ A where F is ij-semi-closed. Then X\A is ij-sg-closed and X\A $\subset$ X\F, where X\F is ij-semi-open set. This implies that ji-scl(X\A) $\subset$ X\F. Now ji-scl(X\A) = X\ji-sint(A). Hence X\ji-sint(A) $\subset$ X\F and F $\subset$ ji-sint(A).

Conversely, if F is an ij-semi-closed set with F $\subset$ ji-sint(A) whenever F $\subset$ A. Then X\A $\subset$ X\F and X\ji-sint(A) $\subset$ X\F. Thus ji-scl(X\A) $\subset$ X\F. Hence X\A is ij-sg-closed and A is ij-sg-open.

Theorem 2.22. If ji-sint(A) $\subset$ B $\subset$ A and A is ij-sg-open, then B is ij-sg-open.

Proof. By hypothesis X\A $\subset$ X\B $\subset$ X\ji-sint(A). Then X\A $\subset$ X\B $\subset$X\[X\ ji-scl(X\A)] = ji-scl(X\A). Thus X\A is ij-sg-closed and hence by Theorem 2.16, B is ij-sg-open.

Lemma 2.23. If A and B are two ij-sg-open subsets of a space X, then A $\cap$ B is ij-sg-open.

Proof. Suppose that F is a ij-semi-closed set contained in A $\cap$ B. Since A and B are ij-sg-open sets, then by Theorem 2.21, F $\subset$ ji-sint(A) and F $\subset$ ji-sint(B). Thus by Lemma 1.2 (iii), F $\subset$ ji-sint(A) $\cap$ ji-sint(B) = ji-sint(A $\cap$ B). Hence F$\subset$ ji-sint(A $\cap$ B) and therefore A $\cap$ B is ij-sg-open.

Definition 2.24. The ij-semi-generalized closure of a subset A of a space X is the intersection of all ij-sg-closed sets containing A and is denoted by ij-sgcl(A).

Lemma 2.25. For any subset A of a space X, A $\subset$ ij-sgcl(A) $\subset$ ji-scl(A) $\subset$ j-cl(A).

Proof. It follows from the facts that every τ_j-closed set is ji-semi-closed and every ji-semi-closed set is ij-sg-closed.

Definition 2.26. A point x of a space X is called an ij-semi generalized limit point (briefly ij-sg-limit point) of a subset A of X, if for each ij-sg-open set U containing x, A $\cap$ U\{x} $\neq \phi$. The set of all ij-sg-limit points of A will be denoted by ij-sgd(A), is called ij-semi-generalized derived set of A.

Lemma 2.27. Let A and B be subsets of a space X. If A $\subset$ B, then ij-sgd(A) $\subset$ ij-sgd(B).

Proof. Obvious.

Lemma 2.28. If A is a subsets of a space X, then ij-sgcl(A) = A $\cup$ ij-sgd(A).

Proof. First we prove that $A \cup$ ij-sgd(A) $\subseteq$ ij-sgcl(A). By Definition 2.26, ij-sgd(A) $\subseteq$ ij-sgcl(A). Since $A \subset$ ij-sgcl(A), then $A \cup$ ij-sgd(A) $\subset$ ij-sgcl(A).

Conversely, suppose that $x \notin (A \cup$ ij-sgd(A)). Then $x \notin A$ and $x \notin$ ij-sgd(A). Since $x \notin$ ij-sgd(A), then there exists an ij-sg-open set U such that $x \in U$ and $A \cap U \setminus \{x\} = \phi$. Since $x \notin A$, then $U \cap A = \phi$. Since $x \notin X \setminus U$ where $X \setminus U$ is ij-sg-closed and $A \subset X \setminus U$. Then $x \notin$ ij-sgcl(A). Hence ij-sgcl(A) $\subset A \cup$ ij-sgd(A) and consequently ij-sgcl(A) = $A \cup$ ij-sgd(A).

Lemma 2.29. A point $x \in$ ij-sgcl(A) if and only if for every ij-sg-open set U containing point x, $U \cup A \neq \phi$.

Proof. Let $x \in$ ij-sgcl(A) and U be an ij-sg-open set containing x. Suppose that $U \cap A = \phi$. Then $A \subset X \setminus U$ where $X \setminus U$ is ij-sg-closed set. Thus $x \in X \setminus U$ which is a contradiction.

Conversely, suppose that for every ij-sg-open set U containing x, $U \cup A \neq \phi$. Let $x \notin$ ij-sgcl(A), then there exists ij-sg-closed F in X such that $A \subset F$ and $x \notin F$. Hence $x \in X \setminus F$ where $X \setminus F$ is ij-sg-open set and $X \setminus F \cap A = \phi$, which is a contradiction.

Theorem 2.30. If A and B are subsets of a space X, then the following are true:

(i) ij-sgd($A \cup B$) = ij-sgd(A) $\cup$ ij-sgd(B)

(ii) ij-sgcl($A \cup B$) = ij-sgcl(A) $\cup$ ij-sgcl(B)

(iii) ij-sgcl(A) = ij-sgcl(ij-sgcl(A)).

Proof. (i) Let A and B be subsets of X. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By Lemma 2.27, and ij-sgd(A) $\subseteq$ ij-sgd($A \cup B$) and ij-sgd(B) $\subseteq$ ij-sgd($A \cup B$). Hence ij-sgd(A) $\cup$ ij-sgd(B) $\subseteq$ ij-sgd($A \cup B$). Conversely, let $x \notin$ ij-sgd(A) $\cup$ ij-sgd(B). Then $x \notin$ ij-sgd(A), $x \notin$ ij-sgd(B) and there exist two ij-sg-open sets U, V such that $x \in U$, $x \in V$, $A \cap U \setminus \{x\} = \phi$ and $B \cap V \setminus \{x\} = \phi$. Hence $x \in U \cap V$, where $U \cap V$ is an ij-sg-open set of X by Lemma 2.23. This implies $(U \cap V) \setminus \{x\} \cap (A \cup B) = \phi$ and $x \notin$ ij-sgd($A \cup B$). Thus ij-sgd($A \cup B$) $\subseteq$ ij-sgd(A) $\cup$ ij-sgd(B) and ij-sgd($A \cup B$) = ij-sgd(A) $\cup$ ij-sgd(B).

(ii) the proof is similar to (i).

(iii) By Lemma 2.25, ij-sgcl(A) $\subseteq$ ij-sgcl(ij-sgcl(A)). Now, let $x \notin$ ij-sgcl(A). This means that by Lemma 2.29, there exists an ij-sg-open set U of X containing x and $U \cap A = \phi$. Suppose that $U \cap$ ij-sgcl(A) $\neq \phi$. Then there is $y \in U \cap$ ij-sgcl(A), so $y \in$ ij-sgcl(A). This implies for every ij-sg-open set V containing y we have $V \cap A \neq \phi$. But U is an ij-sg-open set containing y. Hence $U \cap A \neq \phi$, which is a contradiction. Thus $U \cap$ ij-sgcl(A) $= \phi$ and $x \notin$ ij-sgcl(ij-sgcl(A)). Hence ij-sgcl(A) = ij-sgcl(ij-sgcl(A)).

Definition 2.31. The ij-semi-generalized interior of a subset A of a space X is the union of all ij-sg-open sets contained in A and is denoted by ij-sgint(A).

Lemma 2.32. For any subset A of a space X, we have j-int(A) $\subset$ ji-sint(A) $\subset$ ij-sgint(A).

Proof. The proof follows from the facts that every τ_j-open set is ji-semi-open and every ji-semi-open set is ij-sg-open.

Lemma 2.33. For any subset A of a space X, we have:

(i) ij-sgcl(X\A) = X\ij-sgint(A)

(ii) ij-sgint(X\A) = X\ij-sgcl(A).

Proof. (i) Let $x \notin$ ij-sgcl(X\A), there exists an ij-sg-open set U of X containing x such that $U \cap (X \setminus A) = \phi$. Hence $x \in U \subset A$ and $x \in$ ij-sgint(A). Thus $x \notin$ X\ij-sgint(A).

Conversely, let $x \notin$ X\ij-sgint(A). Thus $x \in$ ij-sgint(A) and there exists an ij-sg-open set U of X such that $x \in U \subset A$. Hence $U \cap (X \setminus A) = \phi$ and $x \notin$ ij-sgcl(X\A).

(ii) The proof is similar to that of (i).

3. Pairwise Generalized Semi-Closed Sets

Definition 3.1. A subset A of a space X is called ij-generalized semi-closed (briefly ij-gs-closed) if ji-scl(A) $\subset$ U whenever $A \subset U$ and U is τ_i-open in X. If $A \subset X$ is 12-gs-closed and 21-gs-closed, then it is said to be pairwise gs-closed (briefly P-gs-closed).

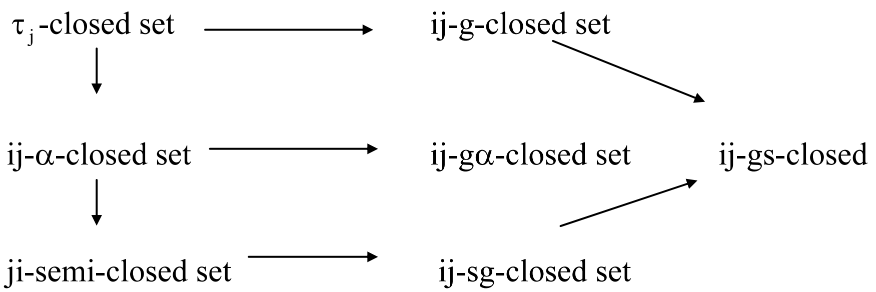
The complement of ij-generalized semi-closed set is called ij-generalized semi-open (briefly ij-gs-open).

Remark 3.2. It follows from Definition 2.1 and 3.1, every ij-sg-closedset is ij-gs-closed. But in Example 2.7, A is 12-gs-closed set and it is not 12-g-closed set.

The following example shows that an ij-gs-closed set need not be ij-sg-closed.

Example 3.3. Let X = {a, b, c, d}, τ_1= {ϕ, {a}, {b}, {a, b}, X} and τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c},{a, b, c}, X}. If A = {a, d}, then A $\subset$ X and 21-scl(A) = {a, c, d}$\subset$ X. So, A is 12-gs-closed set but A is not 12-sg-closed, since 21-scl(A) = {a,c,d}$\not\subset$ {a, d}$\in$ij-SO(X).

From the above discussion and examples we have the following diagram.



Theorem 3.4. A subset A of a bitopological space (X, τ_1, τ_2) is ij-gs-open if and only if F $\subset$ ji-sint(A) whenever F is τ_i-closed and F $\subset$ A.

Proof. It similar to the proof of Theorem 2.21.

Lemma 3.5. If A and B are two ij-gs-open subsets of a space X, then A $\cap$ B is ij-gs-open.

Proof. Similar to the proof of Lemma 2.23.

Lemma 3.6. If A is ij-gs-closed, then ji-scl(A)\A contains no non-empty i-closed set.

Proof. Let F be an i-closed set contained in ji-scl(A)\A. Since A is ij-gs-closed, then ji-scl(A) $\subset$ X\F and so F $\subset$ X\ji-scl(A) and F $\subset$ X\ji-scl (A) $\cap$ ji-scl (A) = ϕ. Hence F = ϕ.

Definition 3.7. The ij-generalized semi-closure of a subset A of a space X is the intersection of all ij-gs-closed sets containing A and is denoted by ij-gscl(A).

Lemma 3.8. Let A be a subset of a space X. Then A ⊂ ij-gscl(A) ⊂ ij-sgcl(A) ⊂ ji-scl(A) ⊂ j-cl(A).

Proof. Follows from Lemma 2.25.

Lemma 3.9. If A is ij-gs-closed set, then A = ij-gscl(A).

Proof. By lemma 3.8, A ⊂ ij-gscl(A). Now, we show that ij-gscl(A) ⊂ A. Since ij-gscl(A) = ∩ { F : A ⊂ F and F is ij-gs-closed in X} and A is ij-gs-closed set, then ij-gscl(A) ⊂ A. Thus A = ij-gscl(A).

The following example shows that ij-gscl(A) needs not be ij-gs-closed.

Example 3.10. Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Let A = {a, b} ⊂ X, then 12-gscl(A) = {a, b} which is 12-gs-closed set. Let B = {a}, then 12-gscl(B) = {a} which is not 12-gs-closed set, since 21-scl(B) = X ⊄ {a} ∈ τ_1.

Definition 3.11. A point x of a space X is called an ij-generalized semi-limit point (briefly ij-gs-limit point) of a subset A of X, if for each ij-gs-open set U containing x, A ∩ U\{x} ≠ ϕ.

The set of all ij-gs-limit points of A will be denoted by ij-gsd(A) and is called the ij-generalized semi-derived set of A.

Lemma 3.12. Let A and B be subsets of a space X. If A ⊂ B, then ij-gsd(A) ⊂ ij-gsd(B).

Proof. Obvious.

Lemma 3.13. If A is a subset of a space X, then ij-gscl(A) = A ∪ ij-gsd(A).

Proof. Similar to the proof of Lemma 2.28.

Lemma 3.14. A point x ∈ ij-gscl(A) if and only if every ij-gs-open set U containing x, U ∩ A ≠ ϕ.

Proof. Similar to the proof of Lemma 2.29.

Theorem 3.15. If A and B are subsets of a space X, then the following are true:

(i) ij-gsd(A $\cup$ B) = ij-gsd(A) $\cup$ ij-gsd(B)

(ii) ij-gscl(A $\cup$ B) = ij-gscl(A) $\cup$ ij-gscl(B)

(iii) ij-gscl(A) = ij-gscl(ij-gscl(A)).

Proof. Similar to the proof of Theorem 2.30. Her we use Lemma 3.5, Lemma 3.8 and Lemma 3.12.

Definition 3.16. The ij-generalized semi-interior of a subset A of a space X is the union of all ij-gs-open sets contained in A and is denoted by ij-gsint(A).

Lemma 3.17. For any subset A of a space X, we have j-int(A) $\subset$ ji-sint(A) ij-sgint(A) $\subset$ ij-gsint(A).

Proof. It follows from the fact that every ij-sg-open set is ij-gs-open set and Lemma 2.32.

Lemma 3.18. For any subset A of a space X, we have:

(i) ij-gscl(X\A) = X\ij-gsint(A)

(ii) ij-gsint(X\A) = X\ij-gscl(A).

Proof. Similar to the proof of Lemma 2.33.

4. Characterization of P-Semi $T_{1/2}$ -Spaces and P-Semi R_0-Spaces

Definition 4.1. A bitopological space (X, τ_1, τ_2) is said to be pairwise semi-T_0 (briefly P-semi T_0) if for each distinct points x,y$\in$X, there exists either an ij-semi-open set containing x but not y or an ij-semi-open set containing y but not x.

Lemma 4.2. Let (X, τ_1, τ_2) be P-semi T_0 if and only if for any points x,y$\in$X such that x $\neq$ y, ji-scl({x}) $\neq$ ij-scl({y}).

Proof. Let X be P-semi T_0 and x,y$\in$X such that x $\neq$ y. Then there exists an ij-semi-open set U such that x $\in$ U and y$\notin$ U. Thus {y}$\cap$ U = ϕ this means that x$\notin$ ij-scl({y}). Since x $\in$ ji-scl({x}), then we have ji-scl({x}) $\neq$ ij-scl({y}).

On the other hand suppose that x,y$\in$X and x $\neq$ y. Then ji-scl({x}) $\neq$ ij-scl({y}). Thus either y$\notin$ ji-scl({x}) or x$\notin$ ij-scl({y}). If y$\notin$ ji-scl({x}), then there exists a ji-semi-open U such that $y \in U$ and $\{x\} \cap U = \phi$, *i.e.* $x \notin U$. If $x \notin$ ij-scl($\{y\}$), then there exists an ij-semi-open V such that $x \in V$ and $\{y\} \cap V = \phi$ or $y \notin V$. In two cases X is P-semi T_0.

Definition 4.3. A bitopological space (X, τ_1, τ_2) is called pairwise semi $T_{1/2}$-space if and only if every ij-sg-closed set is ji-semi-closed.

Theorem 4.4. A space X is P-semi $T_{1/2}$ if and only if every singleton is ji-semi-open or ij-semi-closed.

Proof. Suppose $\{x\}$ is not ij-semi-closed. Then $X \backslash \{x\}$ is ij-sg-closed by Lemma 2.18. Since (X, τ_1, τ_2) is P-semi $T_{1/2}$-space, $X \backslash \{x\}$ is ji-semi-closed and $\{x\}$ is ji-semi-open.

Conversely, let F be ij-sg-closed. For any $x \in$ ji-scl(F), $\{x\}$ is ji-semi-open or ij-semi-closed by assumption.

Case 1. Suppose $\{x\}$ is ji-semi-open. Since $\{x\} \cap F \neq \phi$, then $x \in F$.

Case 2. Suppose $\{x\}$ is ij-semi closed. If $x \notin F$, then this contradicts Lemma 2.19 since $\{x\} \subset$ ji-scl(F)$\backslash$ F. Thus $x \in F$.

From the above two cases we conclude that F is a ji-semi-closed. Hence (X, τ_1, τ_2) is a P-semi $T_{1/2}$-space.

Definition 4.5. A bitopological space (X, τ_1, τ_2) is called pairwise semi-T_1 (briefly P-semi T_1) if for every two distinct points x and y in X, there exists an ij-semi-open set U containing x but not y and an ij-semi-open set V containing y but not x.

Lemma 4.6. A bitopological space (X, τ_1, τ_2) is pairwise semi-T_1 if and only of every singleton is pairwise semi-closed.

Proof. Let (X, τ_1, τ_2) be pairwise semi-T_1. Since for every singleton $\{x\}$ we have $\{x\} \subset$ ij-scl($\{x\}$), then there exists an ij-semi-open set U containing x but $y \notin U$. Thus $\{x\} \cap U \neq \phi$ and $y \notin$ ij-scl ($\{x\}$). Then $\{x\}$ = ij-scl($\{x\}$) and hence $\{x\}$ is ij-semi-closed. Now, for every $x \neq y$, we have $y \in X \backslash \{x\}$. So there exists a ji-semi-open set V_y such that $y \in V_y$ but $x \notin V_y$. Therefore, $y \in V_y \subset X \backslash \{x\}$. Hence $X \backslash \{x\}$ is ji-semi-open and $\{x\}$ is ji-semi-closed.

Conversely, let $\{x\}$ = ji-scl $(\{x\})$ and $x, y \in X$ such that $x \neq y$. Then X\ji-scl($\{x\}$) is a ji-semi-open set containing y but not x. Similarly, if $\{y\}$ = ij-scl$\{y\}$. Then X\ij-scl($\{y\}$) is an ij-semi-open set containing x but not y. Thus X is pairwise semi T_1.

Theorem 4.7. Every P-semi T_1-space is a P-semi $T_{1/2}$-space.

Proof. Let (X, τ_1, τ_2) be P-semi T_1. It suffices to show that a set which is not ji-semi-closed is also not an ij-sg-closed set. Suppose that $A \subset X$ and A is not ji-semi-closed. Let $x \in$ ji-scl(A)\A. Then $\{x\} \subset$ ji-scl(A)\A. Since X is P-semi T_1, then by lemma 4.6, $\{x\}$ is an ij-semi-closed set. By Lemma 2.19, A is not ij-sg-closed.

Example 4.8. Let $X = \{a, b\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, X\}$. Then a space (X, τ_1, τ_2) is 12-semi $T_{1/2}$, but not 12-semi T_1, since $\{a\}$ is 12-semi-closed and is not 21-semi-closed because 21-scl($\{a\}$) $= X \neq \{a\}$.

Definition 4.9. For a subset A of a bitopological space (X, τ_1, τ_2), we define $A^{s\Lambda_{ij}}$ as follows:

$$A^{s\Lambda_{ij}} = \cap \{U: A \subset U, U \in \text{ij-SO}(X)\}.$$

Theorem 4.10. Let A be a subset of a space X . Then $A^{s\Lambda_{ij}} = (A^{s\Lambda_{ij}})^{s\Lambda_{ij}}$

Proof. We have $(A^{s\Lambda_{ij}})^{s\Lambda_{ij}} = \cap\{U: U \in \text{ij-SO}(X), A^{s\Lambda_{ij}} \subset U\} = \cap\{U: U \in \text{ij-SO}(X), (\cap\{V: V \in \text{ij-SO}(X), A \subset V\}) \subset U\} \subset \cap\{U: U \in \text{ij-SO}(X), A \subset U\} = A^{s\Lambda_{ij}}$. This means $(A^{s\Lambda_{ij}})^{s\Lambda_{ij}} \subset A^{s\Lambda_{ij}}$. On the other hand, $A \subset A^{s\Lambda_{ij}}$ for each subset A. Then $A^{s\Lambda_{ij}} \subset (A^{s\Lambda_{ij}})^{s\Lambda_{ij}}$.

Lemma 4.11. A subset A of a space X is ij-sg-closed if and only if ji-scl(A) $\subset A^{s\Lambda_{ij}}$.

Proof. Let $A \subset X$ be an ij-sg-closed set. Suppose that $x \notin A^{s\Lambda_{ij}}$. Then there exists $U \in$ ij-SO(X) such that $x \notin U$ and $A \subset U$. Since A is ij-sg-closed then ji-scl(A) $\subset U$. Hence $x \notin$ ji-scl(A) and ji-scl(A) $\subset A^{s\Lambda_{ij}}$. The proof of the other side follows immediately from Definition 4.9.

Definition 4.12. A bitopological space (X, τ_1, τ_2) is said to be pairwise semi-R_0 (briefly P-semi R_0) if for each $U \in$ ij-SO(X,x), ji-scl($\{x\}$) $\subset$ U.

Theorem 4.13. Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent :

(i) (X, τ_1, τ_2) is pairwise semi R_0-space.

(ii) For any $x \in X$, ij-scl($\{x\}$) $\subset \{x\}^{s\Lambda_{ji}}$.

(iii) For any x, $y \in X$, $y \in \{x\}^{s\Lambda_{ij}}$ if and only if $x \in \{y\}^{s\Lambda_{ji}}$.

(iv) For any x, $y \in X$, $y \in$ ij-scl($\{x\}$) if and only if $x \in$ ji-scl($\{y\}$).

(v) For any ij-semi-closed set F and a point $x \notin F$, there exists a ji-semi-open set U such that $x \notin U$, $F \subset U$.

(vi) Each ij-semi-closed F can be expressed as $F = \cap \{U : F \subset U$, U is ji-semi-open$\}$

(vii) Each ij-semi-open set U can be expressed as the union of ji-semi-closed sets contained in U.

(viii) For each ij-semi-closed set F, $x \notin F$ implies ji-scl($\{x\}$) $\cap F = \phi$.

Proof. (i) $\Rightarrow$ (ii): By Definition 4.9, for any $x \in X$ we have $\{x\}^{s\Lambda_{ji}} = \cap \{U : \{x\} \subset U$, U is ji-semi-open$\}$. Since X is pairwise semi R_0, then each ji-semi-open set U containing x contains ij-scl($\{x\}$). Hence ij-scl($\{x\}$) $\subset \{x\}^{s\Lambda_{ji}}$.

(ii) $\Rightarrow$ (iii): For any x, $y \in X$, if $y \in \{x\}^{s\Lambda_{ij}}$, then $x \in$ ij-scl($\{y\}$). By (ii) since ij-scl($\{y\}$) $\subset \{y\}^{s\Lambda_{ji}}$, we have $x \in \{y\}^{s\Lambda_{ji}}$.

(iii) $\Rightarrow$ (iv): For any x, $y \in X$ if $y \in$ ij-scl($\{x\}$), then $x \in \{y\}^{s\Lambda_{ij}}$. Then by (iii) $y \in \{x\}^{s\Lambda_{ji}}$, and so $x \in$ ji-scl($\{y\}$).

(iv) $\Rightarrow$ (v) Let F be an ij-semi-closed set and a point $x \notin F$. Then for any $y \in F$, ij-scl($\{y\}$) $\subset F$ and $x \notin$ ij-scl($\{y\}$). By (iv), $x \notin$ ij-scl($\{y\}$) and $y \notin$ ji-scl($\{x\}$). Hence there exists a ji-semi-open set U_y such that $y \in U_y$ and $x \notin U_y$. Let $U = \bigcup_{y \in F} \{U_y : y \in U_y$ and $x \notin U_y$, U_y is ji-semi-open$\}$. Then U is a ji-semi-open set such that $x \notin U$ and $F \subset U$.

(v) $\Rightarrow$ (vi): Let F be ij-semi-closed set and suppose that H = $\cap$ {U : F $\subset$ U, U is ji-semi-open}. Then F $\subset$ H and we show that H $\subset$ F. Let x$\notin$ F. Then by (v) there exists a ji-semi-open set U such that x $\notin$ U and F $\subset$ U and hence x$\notin$H. Therefore H $\subset$ F and so F = H.

(vi) $\Rightarrow$ (vii): Obvious.

(vii) $\Rightarrow$ (viii): Let F be an ij-semi-closed set and x $\notin$ F. Then X\F =U is an ij-semi-open set containing x. Then by (vii), there exists a ji-semi-closed set H such that x$\in$H $\subset$ U and so ji-scl({x}) $\subset$ U. Thus ji-scl({x}) $\cap$ F = ϕ.

(viii) $\Rightarrow$ (i): Let U be an ij-semi-open set and x$\in$U. Then x $\notin$X\U which is ij-semi-closed set and by (viii), ji-scl({x}) $\cap$ X\U = ϕ. Thus ji-scl({x}) $\subset$ U. Hence X is pairwise semi R_0.

Definition 4.14. A bitopological space (X, τ_1 , τ_2) is said to be pairwise semi-R_1 (briefly P-semi R_1) if and only if for each x, y$\in$ X such that x$\notin$ij-scl({y}), there exist U$\in$ij-SO(X) and V$\in$ji-SO(X) such that x$\in$ U, y$\in$ V and U $\cap$V = ϕ.

Theorem 4.15. Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent :

(i) X is pairwise semi R_1-space.

(ii) For each x$\in$X, ij-scl({x}) = ij-scl_θ({x}).

(iii) For each ji-semi-compact A $\subset$ X, ij-scl(A) = ij-scl_θ(A).

Proof. (i) $\Rightarrow$ (iii): Generally ij-scl(A) $\subset$ ij-scl_θ(A). Now, let x$\notin$ij-scl(A). For each y$\in$A , x$\notin$ij-scl({y}) and so there exist an ij-semi-open set U_y and a ji-semi-open set V_y such that x$\in U_y$, y$\in V_y$ and $U_y \cap V_y = \phi$. Then {V_y : y$\in$ A} is a ji-semi-open cover of A. Since A is ji-semi-compact, there exists a finite subset A_0 of A such that A $\subset$ $\cup${ V_y : y$\in$ A_0}. Put V = $\cup${ V_y : y$\in A_0$} and U = $\cap$ { U_y : y$\in A_0$}. Then V is a ji-semi-open set, U is an ij-semi-open set such that A $\subset$ V, x$\in$U and U $\cap$V = ϕ. Thus ji-scl(U) $\cap$V = ϕ and so ji-scl(U) $\cap$ A = ϕ. This shows that x$\notin$ ij-scl_θ(A) and therefore ij-scl_θ(A) $\subset$ ij-scl(A).

(iii) $\Rightarrow$ (ii): The proof is obvious, since {x} is ji-semi-compact.

(ii) $\Rightarrow$ (i): Let x$\notin$ij-scl({y}, by (ii) x$\notin$ij-scl_θ({y}). Then there exists an ij-semi-open set U such that x$\in$U and ji-scl(U) $\cap$ {y} = ϕ, Then X\ji-scl(U) is a ji-semi-open set containing y. Also, $U \cap X \backslash$ji-scl(U) = ϕ. This shows that X is pairwise semi R_1-space.

5. An ij-Semi-Generalized Continuous Mappings

Definition 5.1. A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called ij-generalized continuous (briefly ij-g-continuous) if $f^{-1}(V)$ is ij-g-closed in X for every σ_j-closed V of Y.

Theorem 5.2. Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

(i) f is ij-g-continuous

(ii) For each $x \in X$ and for each σ_j-open set V containing f(x), there is an ij-g-open set U containing x such that $f(U) \subset V$.

(iii) f(ij-gcl(A)) $\subset$ j-cl(f(A)) for each subset A of X.

(iv) ij-gcl $(f^{-1}(B)) \subset f^{-1}$(j-cl(B)) for each subset B of Y.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$ and V be σ_j-open set containing f(x). Then by (i), $f^{-1}(V)$ is ij-g-open set of X which containing x. If $U = f^{-1}(V)$, then $f(U) \subset V$.

(ii) $\Rightarrow$ (iii): Let A be a subset of a space X and f(x) $\notin$ j-cl(f(A)). Then there exists σ_j-open set V of Y containing f(x) such that $V \cap f(A) = \phi$. Then by(ii), there is an ij-g-open set U such that $f(x) \in f(U) \subset V$. Hence $f(U) \cap f(A) = \phi$ implies $U \cap A = \phi$. Consequently, $x \notin$ ij-gcl(A) and $f(x) \notin$ f(ij-gcl(A)).

(iii) $\Rightarrow$ (iv): Let B be a subset of Y and $A = f^{-1}(B)$. By (iii) f(ij-gcl $(f^{-1}(B))) \subset$ j-cl$(f(f^{-1}(B))) \subset$ j-cl(B). Thus ij-gcl$(f^{-1}(B)) \subset f^{-1}$(j-cl(B)).

Definition 5.3. A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:

(i) ij-semi-generalized continuous (briefly ij-sg-continuous) if $f^{-1}(V)$ is ij-sg-closed in X for every σ_j-closed V of Y, or equivalently if $f^{-1}(V)$ is ij-sg-open in X for every σ_j-closed V of Y.

(ii)ij-semi-generalized irresolute (briefly ij-sg-irresolute) if $f^{-1}(V)$ is ij-sg-closed in X for every ij-sg-closed set V of Y.

Theorem 5.4. If f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is:

(i) ij-sg-continuous, then f is ij-gs-continuous.

(ii) ji-semi-continuous, then f is ij-sg-continuous.

Proof. (i) Follows from Remark 3.2.

(ii) Follows from Theorem 2.8.

The converse of the above theorem needs not be true as is seen from the following examples.

Example 5.5. Let X =Y= {a, b, c, d}, τ_1 and τ_2 as Example 3.3, σ_1 ={ϕ, {a}, {a, c}, Y} and σ_2 = {ϕ, {b}, {b, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be identity function. Then f is 12-gs-continuous but it is not 12-sg-continuous, since A={a, d} is σ_2–closed set and f^{-1}({a, d}) = {a, d}. By Example 3.3, A is not 12-sg-closed.

Example 5.6. Let X = {a, b, c, d}, τ_1 = {ϕ, {a, d}, X}, τ_2 = {ϕ, {c}, {a, c}, X}, Y = {p, q}, σ_1 ={ϕ, {p}, Y} and σ_2 = {ϕ, {p}, {q}, Y}. Let f: (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a function defined by f(a) = f(b) = f(d) = q and f(c) = p. Then f is 12-sg-continuous but not 21-semi-continuous, since {p} is σ_2 -closed and f^{-1}({p}) = {c}, where {c} is not 21-semi-closed.

Remark 5.7. From above we have the following implications and none of the implications is reversible.

j-continuity ⟶ ji-semi-continuity ⟶ ij-sg-continuity

Theorem 5.8. Let f: (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a function.

(i) f is ij-sg-continuous

(ii) For each $x \in X$ and for each σ_j-open set V containing f(x), there is an ij-sg-open set U containing x such that $f(U) \subset V$.

(iii) $f(\text{ij-sgcl}(A)) \subset \text{j-cl}(f(A))$ for each subset A of X.

(iv) $\text{ij-sgcl}(f^{-1}(B)) \subset f^{-1}(\text{j-cl}(B))$ for each subset B of Y.

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Proof. Similar to the proof of Theorem 5.2.

Theorem 5.9. If f: (X, τ_1, τ_2) → (Y, σ_1, σ_2) is an ij-sg-irresolute and X is P-semi $T_{1/2}$. Then f is ij-irresolute.

Proof. Let V be a ji-semi-closed set of Y. Since V is ij-sg-closed in (Y, σ_1, σ_2) and f is ij-sg-irresolute, then $f^{-1}(V)$ is ij-sg-closed in X. But X is P-semi $T_{1/2}$ and so $f^{-1}(V)$ is ji-semi-closed. Hence f is ji-irresolute.

Theorem 5.10. If f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is ij-irresolute and ji-pre-semi-closed, then for every ij-sg-closed set A of X, f (A) is ij-sg-closed set of Y.

Proof. Let A be an ij-sg-closed set. Suppose that f (A) $\subset$ U, where U is an ij-semi-open in Y. Then A $\subset$ $f^{-1}(U)$ and $f^{-1}(U) \in$ ij-SO(X), since f is ij-irresolute. Since A is ij-sg-closed, ji-scl(A) $\subset$ $f^{-1}(U)$ and hence f(ji-scl(A)) $\subset$ U. Therefore, we have ji-scl(f(A)) $\subset$ ji-scl(f(ji-scl(A))) = f(ji-scl(A)) $\subset$ U, since f is ji-pre-semi-closed. Hence f(A) is ij-sg-closed in Y.

Theorem 5.11. If f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is ij-pre-semi-closed and ji-irresolute, then for every ij-sg-closed set B of Y, $f^{-1}(B)$ is ij-sg-closed set of X.

Proof. Let B be an ij-sg-closed subset of Y and $f^{-1}(B) \subset$ U, where U is a ij-semi-open set of X. Since f is an ij-pre-semi-closed and by Lemma 1.8, there is an ij-semi-open set V such that B $\subset$ V and $f^{-1}(V) \subset$ U. Since B is ij-sg-closed set and B $\subset$ V, then ji-scl(B) $\subset$ V. Hence f^{-1}(ji-scl(B)) $\subset$ $f^{-1}(V) \subset$ U. By Lemma 1.5, ji-scl ($f^{-1}(B)$) $\subset$ U and hence $f^{-1}(B)$ is ij-sg-closed set in X.

Remark 5.12. Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) and g : (Y, σ_1, σ_2) $\rightarrow$ (Z, ν_1, ν_2) be two functions. Then:

(i) If f is ij-sg-irresolute and g is ij-sg-continuous, then gof is ij-sg-continuous.

(ii) If f is ij-sg-continuous and g is j-continuous, then gof is ij-sg-continuous.

(iii) If f and g are both ij-sg-irresolute, then gof is ij-sg-irresolute.

(iv) Let Y be P-semi $T_{1/2}$. If f is ij-irresolute and g is ji-sg-continuous, then gof is ij-semi-continuous.

Proof. (i) Let W be a ν_j-closed set of Z. Since g is an ij-sg-continuous, then $g^{-1}(W)$ is an ij-sg-closed set of Y. Since f is ij-sg-irresolute, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is ij-sg-closed set of X. Hence gof is ij-sg-continuous.

(ii) Let W be a ν_j-closed set of Z. Since g is j-continuous, then $g^{-1}(W)$ is σ_j-closed set of Y. Since f is ij-sg-continuous, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is ij-sg-closed set of X. Hence gof is ij-sg-continuous.

(iii) Let W be an ij-sg-closed set of Z. Since g is ij-sg-irresolute, then $g^{-1}(W)$ is an ij-sg-closed set of Y. Since f is ij-sg- irresolute, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is an ij-sg-closed set of X. Hence gof is ij-sg-irresolute.

(iv) Let W be a ν_i-closed set of Z. Since g is ji-sg-continuous, then $g^{-1}(W)$ is a ji-sg-closed set of Y. Since Y is P-semi $T_{1/2}$, then $g^{-1}(W)$ is ij-semi-closed set. Since f is ij-irresolute, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is ij-semi-closed set of X. Hence gof is ij-semi-continuous.

Remark 5.13. The composition of two ij-sg-continuous functions needs not to be ij-sg-continuous.

Example 5.14. Let X = {a, b, c, d}, Y = {x, y, z}, Z = {p, q}, τ_1= {ϕ, {c}, {b, c}, X}, τ_2 = {ϕ, {c, d}, X}, σ_1 = {ϕ, {z}, Y}, σ_2 = {ϕ, {y, z},Y}, ν_1 = {ϕ, {p}, {q}, Z} and ν_2 = {ϕ, {p}, Z}. Define a function f : (X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) by setting f (a) = f (b) = x and f (c) = y, f(d) = z and a function g : (Y, σ_1, σ_2) $\to$ (Z, ν_1, ν_2) by g(x) = g(y) = q, g(z) = p. Then f and g are 12-sg-continuous. But gof : (X, τ_1, τ_2) $\to$ (Z, ν_1, ν_2) is not 12-sg-continuous, since {q} is ν_2-closed and $(gof)^{-1}(\{q\})$ ={a, b, c}$\notin$12-SGC(X).

Theorem 5.15. If a space X is P-semi $T_{1/2}$ and f: (X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) is bijective, pairwise pre-semi-open, then Y is P-semi $T_{1/2}$.

Proof. Let {y} be a singleton of Y. Since X is P-semi $T_{1/2}$ and f is onto, for some $x \in X$ with f(x) = y, {x} is ji-semi-open or ij-semi-closed by Theorem 4.4. If the singleton {x} is ji-semi-open, since f is pairwise pre-semi-open, then {y} is ji-semi-open. If {x} is ij-semi-closed, then {y} is ij-semi-closed by hypothesis and Lemma 1.6. Thus Y is P-semi $T_{1/2}$, by Theorem 4.4.

Theorem 5.16. If a space X is P-semi $T_{1/2}$ and f: (X, τ_1, τ_2) $\to$ (Y, σ_1, σ_2) is surjective, ji-irresolute and ij-pre-semi-closed, then Y is P-semi $T_{1/2}$.

Proof. Let B be an ij-sg-closed subset of Y. Then by Theorem 5.11, we have $f^{-1}(B)$ is an ij-sg-closed subset of X. It follows by assumptions that, $f^{-1}(B)$ is ji-semi-closed and hence B is ji-semi-closed. It follows that Y is P-semi $T_{1/2}$.

6. An ij- Generalized Semi-Continuous Mappings

Definition 6.1. A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called ij-generalized semi-continuous (briefly ij-gs-continuous) if $f^{-1}(V)$ is ij-gs-closed in X for every σ_j-closed set V of Y, equivalently $f^{-1}(V)$ is ij-gs-open in X for every σ_j-open set V of Y.

Lemma 6.2. (i) Every j-continuous map is ij-g-continuous.

(ii) Every ij-g-continuous map is ij-gs-continuous.

(iii) Every ji-semi-continuous map is ij-gs-continuous.

(iv) If f is ij-sg-continuous, then f is ij-sg-continuous.

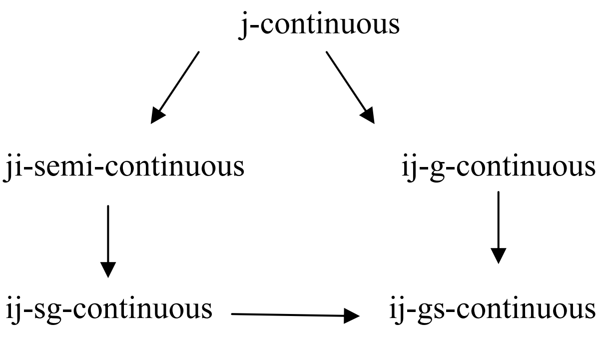
Proof. (i) It follows from the fact that every τ_j-closed is ij-g-cloed set.

(ii) It follows from the fact that every ij-g-closed is ij-gs-closed set.

(iii) It follows from the fact that every ji-semi-closed is ij-gs-closed set.

(iv) It follows from Remark 3.2.

From above we have the following diagram.



Theorem 6.3. Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

(i) f is ij-gs-continuous

(ii) For each $x \in X$ and for each σ_j-open set V containing f(x), there is an ij-gs-open set U containing x such that $f(U) \subset V$.

(iii) $f(ij\text{-}gscl(A)) \subset j\text{-}cl(f(A))$ for each subset A of X.

(iv) $ij\text{-}gscl\,(f^{-1}(B)) \subset f^{-1}(j\text{-}cl(B))$ for each subset B of Y.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. Similar to the proof of Theorem 5.2.

Theorem 6.4. If a function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-continuous and ji-pre-semi-closed, then for every ij-gs-closed set A of X, f (A) is ij-gs-closed set in Y.

Proof. Let A be an ij-gs-closed subset of X. Suppose that $f(A) \subset U$, where U is a σ_i-open set of Y. Then $A \subset f^{-1}(U)$ and $f^{-1}(U)$ is τ_i-open set of X, since f is i-continuous. Since A is ij-gs-closed, then $ji\text{-}scl(A) \subset f^{-1}(U)$ and hence $f(ji\text{-}scl(A)) \subset U$. On the other hand, since f is ji-pre-semi-closed, then $ji\text{-}scl(f(A)) \subset ji\text{-}scl(f(ji\text{-}scl(A))) \subset f(ji\text{-}scl(A)) \subset U$. Hence f(A) is ij-gs-closed in Y.

Theorem 6.5. If a map f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-closed and ji-irresolute, then for each ij-gs-closed set B of Y, $f^{-1}(B)$ is ij-gs-closed in X.

Proof. Let B be an ij-gs-closed subset of Y and $f^{-1}(B) \subset U$, where U is a τ_i-open set of X. Since f is i-closed and by Lemma 1.7, there is a σ_i-open set V such that $B \subset V$ and $f^{-1}(V) \subset U$. Since B is ij-gs-closed set and $B \subset V$, then $ji\text{-}scl(B) \subset V$. Hence $f^{-1}(ji\text{-}scl(B)) \subset f^{-1}(V) \subset U$. By Lemma 1.5, $ji\text{-}scl\,(f^{-1}(B)) \subset U$ and hence $f^{-1}(B)$ is ij-gs-closed set in X.

Remark 6.6. Let f : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and g : $(Y, \sigma_1, \sigma_2) \to (Z, \nu_1, \nu_2)$ be two functions. Then:

(i) If f is ij-gs-continuous and g is j-continuous, then gof is ij-gs-continuous.

(ii) If f is ji-irresolute, i-closed and g is ij-gs-continuous, then gof is ij-gs-continuous.

Proof. (i) Let W be a ν_j-closed set of Z. Since g is j-continuous, then $g^{-1}(W)$ is σ_j-closed set of Y. Since f is ij-gs-continuous, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is ij-gs-closed set of X. Hence gof is ij-gs-continuous.

(ii) Let W be a ν_j-closed set of Z. Since g is an ij-gs-continuous, then $g^{-1}(W)$ is ij-gs-closed set of Y. By Theorem 6.5, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is an ij-gs-closed set of X. Hence gof is ij-gs-continuous .

Definition 6.7. A subset A of a space X is called an ij-interior-closed set (briefly ij-ic-set) if i-int(A) is τ_j-closed.

Lemma 6.8. Let A be a subset of a space X. Then the following are equivalent:

(i) A is ij-ic

(ii) i-int(A) = j-cl(i-int(A)) $\cap$ A.

Proof. (i) $\Rightarrow$ (ii): Let A be a subset of X. We have i-int(A) = F $\cap$ A, where F is τ_j-closed. Then i-int(A) $\subset$ F, this implies j-cl(i-int(A)) $\subset$ F and so i-int(A) = j-cl(i-int(A)) $\cap$ A.

(ii) $\Rightarrow$ (i): Since i-int(A) = j-cl(i-int(A)) $\cap$ A, so that i-int(A) is τ_j-closed. Thus A is ij-ic-set.

Theorem 6.9. If A is a subset of a space X, then A is τ_i-open if and only if A is ij-ic-set and ij-semi-open.

Proof. If A is ij-ic, then by Lemma 6.8, i-int(A) = j-cl(i-int(A)) $\cap$ A. Therefore, if A is ij-semi-open, then A $\subset$ j-cl(i-int(A)), so that j-cl(i-int(A)) $\cap$ A = A. Hence i-int(A) = A and A is τ_i -open.

Conversely, if A is τ_i-open, then A is ij-semi-open. Now, A = j-cl(A) $\cap$ A and i-int(A) = j-cl(i-int(A)) $\cap$ A, since A is τ_i-open. By Lemma 6.8, A is ij-ic-set.

Definition 6.10. A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be ij-ic-continuous if $f^{-1}(V)$ is an ij-ic-set in X, for every σ_i-open set V in Y.

Theorem 6.11. A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is i-continuous if and only if f is ij-semi-continuous and ij-ic-continuous.

Proof. Follows from Theorem 6.9.

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المجموعات الثنائية المغلقة شبه المعممة

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المستخلص. تم استحداث نوعين من المجموعات المغلقة المعممة في الفراغات ثنائية التوبولوجي، وهما المجموعات الثنائية المغلقة شبه المعممة (pairwise semi-generalized closed sets)، والمجموعات الثنائية شبه المغلقة المعممة (pairwise generalized semi closed sets)، كما تم تقديم الخصائص الأساسية لهذه الأنواع المستحدثة، ودرسنا العلاقة بين هذه المجموعات والمجموعات الأخرى، تمت برهنة العديد من النتائج المتعلقة بها، وقُدمت رسوماً توضيحية تبين علاقة كل نوع من هذه المجموعات بالأخرى. كذلك قمنا بتعريف المجموعات الثنائية المعممة المغلقة من النوع α (pairwise generalized α-closed sets)، ومؤثر ij–شبه الإغلاق (ij–شبه الداخلية) المعمم (ij-semi-closure (ij-semi-interior) generalized operator) وكذلك مؤثر ij–الإغلاق (ij–الداخلية) شبه المعمم (ij-closure (ij-interior) semi- generalized operator). كذلك تم تقديم تعريف للمؤثر $A^{s\Lambda_{ij}}$ لمجموعة ما A من الفراغ ثنائي التوبولوجي، وحصلنا على العديد من الخصائص المكافئة للفراغات الثنائية شبه R_0 (pairwise semi R_0–spaces). وباستخدام الأنواع الجديدة من المجموعات المستحدثة تم إدخال توسعات لمفاهيم الانفصال، وأمكن تعريف العديد من مسلمات الانفصال في الفراغات ثنائية التوبولوجي، مثل مسلمات شبه الانفصال $T_{1/2}$

(semi- $T_{1/2}$ separation axioms)، و تم استنتاج أن الفراغ الثنائي شبه T_1 هو ثنائي شبه $T_{1/2}$ ، قُدم مثال عكس يوضح أن العكس لا يكون صحيحاً دائماً. بالمقابل وباستخدام هذين النوعين من المجموعات، تم إدخال أنواع مختلفة من الرواسم المتصلة المعممة، والرواسم المترددة المعممة، وقد دُرست خصائصها وعلاقتها بالأنواع الأخرى من الرواسم.