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Optimum Delta-V Maneuver for Refueling Space Stations

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Abstract. In this paper, an algorithm for optimum velocity increment maneuver (optimum delta-V) for refueling space stations will be established. Numerical applications of the algorithm are also given.

1. Introduction

The engineering feasibility of a space mission is overwhelmingly dictated by the amount of propellant required to accomplish it. The need for lowering the fuel consumption is so important for space mission guidance[1]and navigation[2]. Fuel in space continues to be extraordinarily expensive thereby dictating the feasibility of any proposed architecture[3]. It is worth noting that since current launch costs continue to be high ,the economics of refueling an aging spacecraft need to be offset by possibility of launching a cheaper, advanced spacecraft. As a result of this economic fact, the refueling should be simply to break even[4].

In other words, the refueling should be performed with minimum fuel trajectories to the space station, to this goal the present paper is devoted, for which we shall establish an algorithm for optimum velocity increment maneuvers for refueling space stations.

2. Basic Formulations

Definition of the Problem

The process of departing from a state on a specified orbit and attaining a desired position of the space station is called *interception*[5]. A typical intercept maneuver is illustrated in Fig. 1.

Minimum Velocity Increment Intercept Mode

Minimum velocity increment intercept mode is given in details in reference[6], in what follows, we shall summarize its basic formulations through three steps.

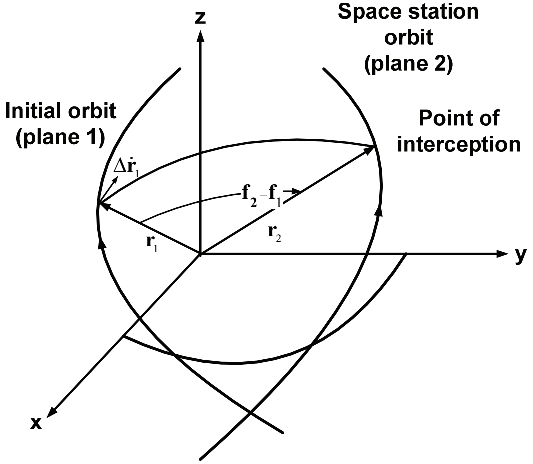

Fig. 1. Intercept maneuver.

First Step

Is to express the vehicle velocity vector $\dot{\mathbf{r}}_1$ at the first terminal in terms of the *unknown* orbital parameter p and the *known* position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and the true anomaly difference $\Delta f = (f_2 - f_1)$ between them. In this respect we have:

$$\dot{\mathbf{r}}_1 = \left(\omega_1\, p^{-1/2} + p^{1/2}\,\omega_2\right)\mathbf{U}_1 + \left(\omega_3 p^{1/2}\right)\mathbf{U}_2 \tag{1}$$

with

$$\omega_1 = \sqrt{\mu}\left\{\frac{1-\cos(f_2 - f_1)}{\sin(f_2 - f_1)}\right\}, \tag{2}$$

$$\omega_2 = \frac{-\sqrt{\mu}}{r_2 \sin(f_2 - f_1)}, \tag{3}$$

$$\omega_3 = \frac{\sqrt{\mu}}{r_1 \sin(f_2 - f_1)}, \tag{4}$$

where $\mathbf{U_1}$ $\left(\mathbf{U_2}\right)$ is the unit vector from the dynamic centre which points to the vehicle at the first (second) terminal and μ is the gravitational parameter. The values of U_i are given by the following expression:

$$\mathbf{U}_i = \frac{\mathbf{r}_i}{|\mathbf{r}_i|} : i = 1,2 \tag{5}$$

Second Step

It is desired to find the orbital parameter p that minimizes the velocity increment ΔV_1 at the first terminal, and this could be obtained as follows:

$$\eta_1 s^4 + \frac{1}{2}\eta_2 s^3 - \frac{1}{2}\eta_3 s - \eta_4 = 0 \quad \eta_1 s^4 + \frac{1}{2}\eta_2 s^3 - \frac{1}{2}\eta_3 s - \eta_4 = 0$$

Since ΔV_1 is given as:

$$\Delta V_1 = [\left\langle \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_1^*, \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_1^* \right\rangle]^{1/2} \tag{6}$$

where $\dot{\mathbf{r}}_1^*$ is the known velocity vector before the application of the targeting velocity increment, $|\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_1^*|$ and $\left\langle \mathbf{A}, \mathbf{B} \right\rangle$ is the scalar product of the two vectors **A** and **B.** Utilizing Equation (1), by direct substitution and expansion then by differentiating the resulting equation with respect to p and equating to zero we get the quartic

$$\eta_1 s^4 + \frac{1}{2}\eta_2 s^3 - \frac{1}{2}\eta_3 s - \eta_4 = 0 \tag{7}$$

where

$$\eta_1 \equiv \omega_2^2 + \omega_3^2 + 2\omega_2\omega_3 \left\langle \mathbf{U}_1, \mathbf{U}_2 \right\rangle , \tag{8}$$

$$\eta_2 \equiv -2\omega_2 \left\langle \dot{\mathbf{r}}_1^*, \mathbf{U}_1 \right\rangle - 2\omega_3 \left\langle \dot{\mathbf{r}}_1^*, \mathbf{U}_2 \right\rangle , \tag{9}$$

$$\eta_3 \equiv -2\omega_1 \left\langle \dot{\mathbf{r}}_1^*, \mathbf{U}_1 \right\rangle , \tag{10}$$

$$\eta_4 \equiv \omega_1^2 , \tag{11}$$

$$\eta_5 \equiv \left\langle \dot{\mathbf{r}}_1^*, \dot{\mathbf{r}}_1^* \right\rangle + 2\omega_1\omega_3 \left\langle \mathbf{U}_1, \mathbf{U}_2 \right\rangle + 2\omega_1\omega_2 , \tag{12}$$

$$s = p^2 . \tag{13}$$

Solution of Equation (7) for the roots s_i enable us to determine p_i. Compute $\dot{r}_{1,i}$ from Equation (1) as

$$\dot{r}_{1,i} = (\omega_1 p_i^{-1/2} + \omega_2 p_i^{1/2})\mathbf{U}_1 + (\omega_3 p_i^{1/2})\mathbf{U}_2 \ . \tag{14}$$

The required minimum of the velocity increment ΔV_1 is

$$\Delta V_1 = \text{Minimum}[\Delta V_{1,i} = |\,\dot{\mathbf{r}}_{1,i} - \dot{\mathbf{r}}_1^*\,|\,] \quad \forall i \tag{15}$$

Third Step

This step is devoted to determine the minimum velocity increment intercept orbit and this can be obtained easily as follows**:**

♦ Since Equation (14) has produced the appropriate velocity vector $\dot{\mathbf{r}}_1$.

♦ Since the position vector $\mathbf{r}_1$ is already known, then utilizing the standard method[5] of orbit determination to find an orbit from $\mathbf{r}$ and $\dot{\mathbf{r}}$.

3. Computational Developments

Computational Algorithm

● *Purpose***:** To compute a(semi major axis of the orbit), e(the eccentricity of the orbit), p and Δv of all possible orbits between the terminals $\mathbf{r}_1$ and $\mathbf{r}_2$ that give minimum velocity intercept trajectory.

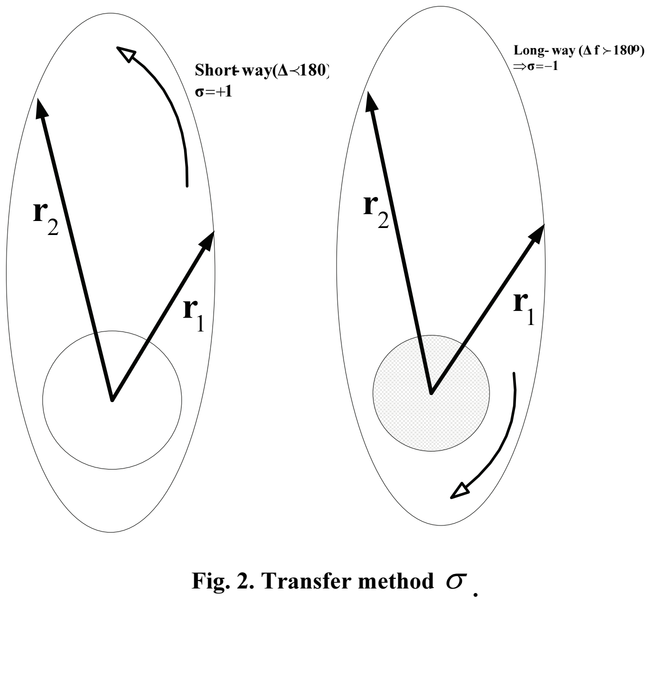

Fig. 2. Transfer method σ .

• *Input*: $\mathbf{r}_1(x_1, y_1, z_1)$; $\mathbf{r}_2(x_2, y_2, z_2)$; $\dot{r}_1^*(\dot{x}_1^*, \dot{y}_1^*, \dot{z}_1^*)$; μ ; σ

• ***Computational Sequence***

1- $r_i = |\mathbf{r}_i| \quad ; i = 1,2$

2- $\mathbf{U}_i = \mathbf{r}_i / r_i \ ; i = 1,2$

3- $C_{21} = \langle \mathbf{U}_1, \mathbf{U}_2 \rangle$

4- $S_{21} = \sigma \times \sqrt{1 - C_{21}^2}$

5- $Q_1 = \langle \mathbf{U}_1, \dot{\mathbf{r}}_1^* \rangle$; $Q_2 = \langle \mathbf{U}_2, \dot{\mathbf{r}}_1^* \rangle$; $Q_3 = \langle \dot{\mathbf{r}}_1^*, \dot{\mathbf{r}}_1^* \rangle$

6- $\omega_1 = \dfrac{\sqrt{\mu}(1 - C_{21})}{S_{21}}$; $\omega_2 = -\dfrac{\sqrt{\mu}}{r_2\, S_{21}}$; $\omega_3 = \dfrac{\sqrt{\mu}}{r_1\, S_{21}}$

7- $\eta_1 = \omega_2^2 + \omega_3^2 + 2\omega_2\omega_3 C_{21}$; $\eta_2 = -2\omega_2 Q_1 - 2\omega_3 Q_2$;

$\eta_3 = -2\omega_1 Q_1$; $\eta_4 = 2\omega_1^2$;

$\eta_5 = Q_3 + 2\,\omega_1\,\omega_3\,C_{21} + 2\,\omega_1\,\omega_2$

8- Solve the quartic:

$$\eta_1\, s^4 + \eta_2\, s^3 + \eta_3\, s^2 + \eta_4\, s + \eta_5 = 0$$

for the real positive (if exist) roots s_i

9- Determined the number of these real positive roots, let it be N

10- If N=0, print "All roots of the quartic are complex $\Rightarrow$ No solution to the problem ", go to step **20**

11- For these real positive roots s_i compute $p_i = \sqrt{s_i}\, ; i = 1(1)N$

12- For $i = 1(1)\,N$ perform steps **13** to **17**

13- $p = p_i$

14- $\beta_k = p / r_k - 1\, ; k = 1,2$

15- $\gamma_1 = (C_{21}\beta_1 - \beta_2) / S_{21}$; $\gamma_2 = (-C_{21}\beta_2 + \beta_1) / S_{21}$

16- $e = (\gamma_1^2 + \beta_1^2)^{1/2}$

17- $a = p/(1-e^2)$

18- $\dot{\mathbf{r}}_{1,i} = \left(\omega_1\, p_i^{-1/2} + p_i^{1/2}\,\omega_2\right)\mathbf{U}_1 + \left(\omega_3 p_i^{1/2}\right)\mathbf{U}_2$

19- $\Delta V_i = \left|\, \dot{\mathbf{r}}_{1,i} - \dot{\mathbf{r}}^*_{1,i} \right|$

20- End

Numerical Applications

All the distances are expressed in terms of the Earth's radius (ER = 6378. 1363 km), while the velocities are in terms of $\mathrm{ER/UT}(\equiv 7.90536\ \mathrm{km/sec})$. With this canonical units $\mu = 1$. Also ΔV in km/sec.

● *Cases*

in what follows we shall consider five cases for testing the code of the above algorithm.

Case 1	$\mathbf{r}_1 \equiv (2.5,0.0,0.0)$; $\mathbf{r}_2 \equiv (1.915111,\ 1.606969\ ,\ 0.0)$; $\dot{\mathbf{r}}^* \equiv (0.6\ ,0.8\ ,0.0)$
Case 2	$\mathbf{r}_1 \equiv (2.5,\ 1.0,\ 0.0)$; $\mathbf{r}_2 \equiv (3.0,\ 2.0\ ,\ 0.0)$; $\dot{\mathbf{r}}^* \equiv (1.6\ ,\ 2.8\ ,\ 3.0)$
Case 3	$\mathbf{r}_1 \equiv (-.9,\ .3,\ .8)$; $\mathbf{r}_2 \equiv (.33,\ .8\ ,\ -4.0)$; $\dot{\mathbf{r}}^* \equiv (.53\ ,.23\ ,.08)$
Case 4	$\mathbf{r}_1 \equiv (123,\ 22\ ,808)$; $\mathbf{r}_2 \equiv (33,\ 8\ ,-4)$; $\dot{\mathbf{r}}^* \equiv (1.2,\ -2.3\ ,.08)$
Case 5	$\mathbf{r}_1 \equiv (10,\ 20\ ,30)$; $\mathbf{r}_2 \equiv (1,\ 19\ ,1)$; $\dot{\mathbf{r}}^* \equiv (1.,23,\ 25)$

● *The Real Positive Solutions of Quartic*

Case 1	$s_1 = 1.83176$; $s_2 = 2.78502$
Case 2	$s_1 = 1.80362$; $s_2 = 11.6246$
Case 3	$s_1 = 0.402306$
Case 4	$s_1 = 1.81214$; $s_2 = 8.21447$
Case 5	No real positive solutions of quartic No solution to the problem

Corresponding to the solutions of the quartic we have the following possible orbits

Case 1

Orbit	1	p	1.35343	e	0.488063	a	1.77663	V	6.42261
Orbit	2	p	1.66884	e	0.353802	a	1.90763	V	4.70494

1- Both of the two orbits are elliptic

2- The optimal orbit with the least value of ΔV is the second orbit

Case 2

Orbit	1	p	1.34299	e	0.832932	a	4.38564	V	29.8905
Orbit	2	p	3.40948	e	1.55167	a	˜ 2.42208	V	22.5007

1- One of the possible orbits is elliptic, while the other is hyperbolic

2- The optimal orbit with the least value of ΔV is the second orbit(hyperbolic)

Case 3

Orbit	1	p	0.634276	e	1.58709	a	˜ 0.417598	V	27.1859

Only one possible hyperbolic orbit

Case 4

Orbit	1	p	1.34616	e	1.03769	a	17.5268	V	8.58544
Orbit	2	p	2.86609	e	1.01218	a	˜ 116.955	V	9.52117

1- Both of the two orbits are hyperbolic

2- The optimal orbit with the least value of ΔV is the first orbit

In concluding the present paper, an effective algorithm is established for optimum velocity increment maneuver (optimum delta-V) for refueling space stations. Numerical applications of the algorithm are also given.

Acknowledgment

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مناورة مثالية لـــ ΔV لتَزْويد المحطات الفضائية بالوقود

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المستخلص. تم في هذا البحث تشييد خوارزميـــة فعّالـــة للمنـــاورة المثالية لــــ ΔV وذلك لتَزْويد المحطات الفضائية بـــالوقود كمـــا أعطيت بعض التطبيقات العددية لهذه الخوارزمية.