

Studies on Quantum Gravity

I. Analytical Model of the Ideal Universe with Zero Temperature Background

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Abstract. In this paper of the series, we set the background to absolute zero, we arrive at the Ideal Universe with literal analytical solutions in power series forms. These solutions are then developed for the radius of curvature and the expansion velocity for the Ideal Universe with an absolute zero temperature background. For computational developments of these solutions, an efficient method using continued fraction theory is provided together with some numerical examples.

1. Introduction

Ideally all objects that are above absolute zero temperature emit energy^[1-4]. This energy is emitted from an energetic object and then absorbed by the less energetic object ultimately ending up, what we will call “domino effect”, in the surrounding background medium/reservoir which has the least amount of energy. Ideally this background medium plays the most important role which initiates the direction of energy transfer and the arrow of Time^[4]. All astronomical observations reveal that our current universe has a homogeneous and isotropic non-zero temperature background medium filling all of space^[4-8]. Since the background temperature determines the direction of energy transfer and the arrow of time, conclusions are made; we must construct a Universe with an absolute zero temperature background, known as The Ideal Universe, to be an absolute frame of reference with that of the known Universe.

Studying The Ideal Universe, the evolution of its scale factor, the behaviour of the scale factor, and how it increases or decreases with Time, one can precisely calculate expansion or contraction of the Universe. For such studies,

the differential system of The Ideal Universe, together with the two free parameters, the density and acceleration parameters are usually used.

In the present paper, literal analytical solutions in power series forms are developed for the radius of curvature and the expansion velocity of the zero temperature cosmological model of the universe. For computational developments of these solutions, an efficient method using continued fraction theory is provided together with some numerical examples.

Undoubtedly, it is true that numerical integration methods can provide accurate models, it is most certain on the other hand that if full analytical formulae are utilized with nowadays existing symbols used for manipulating digital computer programs, they definitely become invaluable for obtaining models with desired accuracy. Moreover, these analytical formulae usually offer much deeper insight into the nature of a model as compared to numerical integration. By proposing The Ideal Universe with analytical formulation, the beauties of the regularity of the recursive^[9] computations facilitate precisely the algorithms.

2. Einstein Field Equations

2.1 General Equations

• The three general equations from Einstein theory of general relativity^[6,10-12] that describing the structure and evolution of a spherically symmetric universe are:

$$3 \frac{\dot{R}^2}{R^2} + \frac{3kc^2}{R^2} = 8\pi G\rho + \Lambda c^2, \quad (1)$$

$$2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = -\frac{8\pi GT}{c^2} + \Lambda c^2, \quad (2)$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3} \left[\rho + \frac{3T}{c^2} \right], \quad (3)$$

where

$R(t)$ = radius of curvature of the universe,

k = curvature index = 0, ± 1 ($k = 1$,elliptical closed space ; $k=0$,Euclidean flat space;

$k = -1$,hyperbolic open space),

ρ = mean density of matter and energy,

Λ = cosmological constant,

T = temperature of matter and radiation,

G = gravitational constant.

• Two free parameters are usually introduced to determine the physical characteristics of a particular model of the universe:

1-The density parameter σ_0 :

$$\sigma_0 = \frac{4}{3} \pi \rho_0 G H_0^{-2} \tag{4}$$

2-The acceleration parameter q_0 :

$$q_0 = - \frac{\ddot{R}_0}{R_0} H_0^{-2} , \tag{5}$$

where ,the subscript zero stands for the physical quantities at their present value and

$$H_0 = \frac{\dot{R}_0}{R_0} \tag{6}$$

is the Hubble constant.

• These two parameters are linked with the classical parameters by the following relations :

$$\Lambda c^2 = 3H_0^2 (\sigma_0 - q_0) \tag{7}$$

$$R_0^2 = k c^2 (3\sigma_0 - q_0 - 1)^{-1} H_0^{-2} \tag{8}$$

$$k = \text{Sign}(3\sigma_0 - q_0 - 1) \tag{9}$$

where the Sign(x) is zero, + 1 or - 1 for x, zero, positive or negative.

2.2 Physical Characteristics and the Classical Parameters

• The relations between the physical characteristics of a model for the universe and the classical parameters are summarized as^[6,10-12]:

▲ If $\Lambda = 0$

(a) $q_0 > 0.5 \Leftrightarrow k > 0 \Leftrightarrow$ (positive curvature "closed" universe) \Leftrightarrow universe will eventually recontract

(b) $q_0 = 0.5 \Leftrightarrow k = 0 \Leftrightarrow$ (zero curvature "flat" universe) \Rightarrow universe will expand forever

(c) $q_0 < 0.5 \Leftrightarrow k < 0 \Leftrightarrow$ (negative curvature "open" universe) \Rightarrow universe will expand forever

▲ If $\Lambda \neq 0$

(a) $\sigma_0 > \frac{1}{3}(q_0 + 1) \Leftrightarrow k > 0$ (positive curvature "closed" universe) and in this case,

$$\sigma_0 - q_0 \geq \frac{1}{\sigma_0^2} \left(\sigma_0 - \frac{q_0 + 1}{3} \right)^3 \Leftrightarrow \text{universe will expand forever}$$

$$\sigma_0 - q_0 < \frac{1}{\sigma_0^2} \left(\sigma_0 - \frac{q_0 + 1}{3} \right)^3 \Leftrightarrow \text{universe will eventually recontract}$$

(b) $\sigma_0 = \frac{1}{3}(q_0 + 1) \Leftrightarrow k = 0$ (zero curvature "flat" universe) and in this case,

$$\sigma_0 \geq q_0 \Leftrightarrow \text{universe will expand forever}$$

$$\sigma_0 < q_0 \Leftrightarrow \text{universe will eventually recontract}$$

(c) $\sigma_0 < \frac{1}{3}(q_0 + 1) \Leftrightarrow k < 0$ (negative curvature "open" universe) and in this case,

$$\sigma_0 \geq q_0 \Leftrightarrow \text{universe will expand forever}$$

$$\sigma_0 < q_0 \Leftrightarrow \text{universe will eventually recontract.}$$

3. Zero Temperature Models

3.1 Basic Equation

In the present paper we assume that the temperature is zero, a condition justified by the observations^[1-4]. The basic differential equation for these models could be obtained from the above equations as follows

- Put $T = 0$ in Equation (2) then we get from Equation (1) :

$$\ddot{R} = -\frac{4\pi G\rho R}{3} + \frac{R\Lambda c^2}{3}$$

- The conditions at which the total amount of mass has to remain constant during the evolution of the universe is

$$\rho_0 R_0^3 = \rho(t)R(t) ,$$

using this condition into the above equation we get

$$\ddot{R} = -4\pi G\rho_0 \frac{R_0^3}{3R^2} + \frac{\Lambda c^2 R}{3}, \quad (10)$$

this is the required equation for the zero temperature cosmological model of the universe. In this equation:

▲ The first term represents the gravitational force working to keep the universe together. As usual, its value decreases with the square of the distance.

▲ The second term of the equation is the cosmological force. If Λ is zero, gravitation is the only force. If not, the cosmological force is directed outward if Λ is positive and inwards in negative.

3.2 Transformed Equation

- Usually Equation (10) is transformed into a form that contains only σ_0 and q_0 as free parameters .This could be performed by changing both the dependent and the independent variables as follows:

- ▲ The dependent variable R is changed to

$$R = Y R_0 \quad (11)$$

that to say ,the scale factor is taken as unit of length.

▲ The independent variable t is changed to

$$t = XH_0^{-1} \quad (12)$$

that is to say H_0^{-1} is taken as unit of time, for example if we used for H_0 the value of 55km / sec/ Mpc ,then the unit of time is 1.801×10^{10} years

▲ Let " dash" denotes the differentiation with respect to X ,so

$$\dot{R} = H_0 R' \quad \text{and} \quad \ddot{R} = H_0^2 R''$$

then Equation(10) reduces by the aid of Equations (4),(7)and(11) to

$$Y'' = -\frac{\sigma_0}{Y^2} + (\sigma_0 - q_0)Y. \quad (13)$$

This is the required transformed equation, which is free from Hubble constant and to be solved with the initial conditions:

$$\text{at } X = 0, \quad Y = 1 \quad \text{and} \quad Y' = 1 \quad (14)$$

3.3. Linear System

The linear system of the zero temperature models could be obtained by letting

$$S = \frac{1}{Y^2}, \quad (15)$$

then Equation(13) is transformed into the linear differential system:

$$Y' = Z \quad (16)$$

$$Z' = -\sigma_0 S + (\sigma_0 - q_0)Y \quad (17)$$

$$S'Y = -2ZS \quad (18)$$

which is to be solved subject to the initial conditions :

$$\text{At } X = 1 \quad ; \quad Y = Z = S = 1 \quad (19)$$

4. Analytical Solution

• Undoubtedly true that, in the absence of closed analytical solution of a given differential system the power series solution (which of course assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is no less (and some – times even more) convenient than the elementary functions especially when the problems are to be studied on computers. In fact, most computers often use series in the calculations of the majority of the elementary functions.

• The power series solution of the above system{Equations(16),(17) and(18)} which satisfies the intial conditions of Equation (19) could be obtained through the follwing steps:

Step 1: Power series for the variables

Let

$$Y = \sum_{n=1}^{\infty} y_n X^{n-1} \tag{20}$$

$$Z = \sum_{n=1}^{\infty} z_n X^{n-1} \tag{21}$$

$$S = \sum_{n=1}^{\infty} s_n X^{n-1} \tag{22}$$

Step 2: Product of two infinit power series

Let W, U be two infinit power series of the form:

$$W = \sum_{i=1}^{\infty} w_i X^{i-1}, \quad U = \sum_{i=1}^{\infty} u_i X^{i-1},$$

then

$$F = W \times U = \sum_{n=1}^{\infty} f_n X^{n-1} \tag{23}$$

where

$$f_n = \sum_{k=1}^n u_k w_{n-k+1}; \quad \forall n \geq 1 \tag{24}$$

Step 3: Substitution step.

Substituting Equations (20),(21) and (22) into Equations (16),(17) and (18) we get with the aid of Equations (23) and (24) we get

$$\sum_{n=1}^{\infty} n y_{n+1} X^{n-1} = \sum_{n=1}^{\infty} z_n X^{n-1} \quad (25)$$

$$\sum_{n=1}^{\infty} n z_{n+1} X^{n-1} = -\sigma_0 \sum_{n=1}^{\infty} s_n X^{n-1} + (\sigma_0 - q_0) \sum_{n=1}^{\infty} y_n X^{n-1} \quad (26)$$

$$\sum_{n=1}^{\infty} \left[\sum_{k=1}^n k s_{k+1} y_{n-k+1} \right] X^{n-1} = -2 \sum_{n=1}^{\infty} \left[\sum_{k=1}^n z_k s_{n-k+1} \right] X^{n-1} \quad (27)$$

Step 4: Identification of equal powers of X .

Equating the coefficients of equal powers of X in both sides of Equations (25),(26) and(27) we get the recursion equations

$$n y_{n+1} = z_n \quad (28)$$

$$n z_{n+1} = -\sigma_0 s_n + (\sigma_0 - q_0) y_n \quad (29)$$

$$n s_{n+1} = -2 \sum_{k=1}^n z_k s_{n-k+1} - \sum_{k=1}^{n-1} k s_{k+1} y_{n-k+1} \quad (29)$$

Step 5: Initial conditions.

From Equations (19),(28),(29) and(30) we get

$$y_1 = z_1 = s_1 = 1 \quad (31)$$

$$y_2 = 1 ; z_2 = -q_0 ; s_2 = -2 \quad (32)$$

These equations are applied $\forall n = 2, 3, \dots, n_t$, where n_t is the number of terms of the power series. If Equations (28),(29) and(30) are used in the same order as they stand, then all the coefficients of the power series are completely determined in a full recursive way. Finally, It should also be noted that, these series could be used for future ($X > 0$) as well as for past ($X < 0$).

Now, for given values of the parameters σ_0, q_0 and a value of X one can use series (19) and (20) to compute, the radius of curvature of the universe $R(t)$ ($= R_0 Y$) and its

expansion velocity $\dot{R}(t)$ ($= R_0 Z H_0$) at a any time t ($= X H_0^{-1}$). In what follows we shall consider the reverse of the above problem .

5. Inverse Solution

5.1 Series Inversion of Power Series

• General algorithm for reversing a power series will be developed as follows. Consider the functional equation

$$\eta = \zeta + \alpha \phi(\eta), \quad |\alpha| < 1, \tag{33}$$

then according to Lagrange expansion theorem ^[13], we have

$$\eta = \zeta + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} [\phi(\zeta)]^n. \tag{34}$$

• Let $h(\theta)$ be a function which can be expressed in a Taylor series in the neighborhood of $\theta = \theta_0$. Thus

$$h(\theta) = h_0 + \sum_{j=1}^{\infty} \frac{B_j}{j!} (\theta - \theta_0)^j, \tag{35}$$

where

$$B_j = \left. \frac{d^j h(\theta)}{d\theta^j} \right|_{\theta = \theta_0}$$

In the following ,we assume that B_1 is different from zero and write Equation (35) in the form

$$\theta = \theta_0 + (h - h_0)\phi(\theta), \tag{36}$$

where $\phi(\theta)$ is defined by

$$\phi(\theta) = \frac{1}{B_1 + \sum_{j=1}^{\infty} [B_{j+1} / (j+1)!] (\theta - \theta_0)^j}. \tag{37}$$

Equation (36) is precisely the form as Equation (33), then we can express θ as a power series in $\alpha = h - h_0$ to get

$$\theta(h) = \theta_0 + \sum_{n=1}^{\infty} \frac{C_n}{n!} (h - h_0)^n, \tag{38}$$

where

$$C_n = \frac{d^{n-1}}{d\theta^{n-1}} [\phi(\theta)]^n \Big|_{\theta = \theta_0} \tag{39}$$

and $\phi(\theta)$ is defined in Equation (37). The series for $\theta(h)$ is said to be the reverse of the series for $h(\theta)$.

Battin^[14] developed an elegant algorithm to express n of the coefficients C_1, C_2, \dots of the reversed series in terms of the coefficients B_1, B_2, \dots of the original series. The basic equations of this algorithm are

$$D_0^1 |_{\theta=\theta_0} = \phi^{(0)}(\theta_0) \equiv \phi_0^{(0)} \equiv \frac{1}{B_1}, \tag{40}$$

$$\frac{d^k \phi(\theta)}{d\theta^k} \equiv \phi_0^{(k)} = -\frac{1}{B_1} \sum_{i=1}^k \frac{1}{i+1} \binom{k}{i} B_{i+1} \phi_0^{(k-i)}, \quad k = 1, 2, \dots, n-1, \tag{41}$$

$$D_k^n = \frac{d^k}{dx^k} [\phi(\theta)]^n = n \sum_{j=0}^{k-1} \binom{k-1}{j} D_j^{n-1} \phi_0^{(k-j)}, \tag{42}$$

$$C_n = D_{n-1}^n |_{x=x_0}. \tag{43}$$

5.2 The Inverse Zero Temperature Model

The inverse problem of the zero temperature cosmological model is to find the time $t = \tilde{t}$ (say) [or equivalently the value $X = \tilde{X}$] at which the radius of curvature of the model $R = \tilde{R}$ [or equivalently at the value $Y = \tilde{Y}$]. The following is an algorithm to illustrate the solution of this problem.

5.2.1 Algorithm 1

- Purpose: to find \tilde{t}

- Input : \tilde{Y}, H_0, n_t and $y_k ; k = 1, 2, \dots, n_t$

- Computational steps

1- Set $B_{i+1} = (i + 1)! y_{i+2} \forall i = 0, 1, \dots, m (= n_t - 2)$

2- Apply the algorithm of Subsection 5.1 to find the β 's coefficients in the expansion.

$$\tilde{X} = \sum_{k=1}^{m_t} \beta_k \frac{(\tilde{Y} - 1)^k}{k!}$$

3- $\tilde{t} = \tilde{X} / H_0$

4- End

The above algorithm could also be used to find the time at which the expansion velocity \tilde{R} [or equivalently at the value $Z = \tilde{Z}$] of the model is given. This could be done by replacing y 's by z 's and \tilde{Y} by \tilde{Z} in the computational steps of algorithm 1

To this end, it should be mentioned that ,the convergence of the fundamental power series are not studied in the present paper and will constitute a task to which we shall consider latter. However we can in general improve the convergence of a power series by means of the known Euler's device.

6. Computational Developments

In fact, continued fraction expansions are, generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series. Due to the importance of accurate evaluations and the efficiency of continued fractions, I purpose to use them as the computational tools for evaluating zero temperature cosmological model . To do so, two steps are to be performed:

- (1) Transform the given power series into continued fraction (Section 6.1).
- (2) Evaluating the resulting continued fraction (Section 6.2), as follows:

6.1 Euler's Transformation

Generally an infinite series (a power series is special case of it) of functions could be converted into a continued fraction according to Eulers transformation^[14] which is

$$\sum_{k=0}^{\infty} U_k \equiv \frac{n_1}{d_1 + \frac{n_2}{d_2 + \frac{n_3}{d_3 + \frac{n_4}{\dots}}}}} \equiv \frac{n_1}{d_1 + \frac{n_2}{d_2 + d_3 + d_4 + \dots}} + \dots$$

where

$$n_1 = U_0 ; n_2 = U_1 ; n_i = -U_{i-1} \times U_{i-3}, \forall i \geq 3$$

$$d_1 = 1 ; d_j = U_{j-2} + U_{j-1} \forall j \geq 2.$$

6.2 Top-Down continued Fraction Evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method, obviously is having to decide far down the fraction to being in order to ensure convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi^[9] proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as

$$q = \frac{n_1}{d_1 + \frac{n_2}{d_2 + \frac{n_3}{d_3 + \dots}}}$$

then initialize the following parameters

$$a_1 = 1 , b_1 = n_1 / d_1 , q_1 = n_1 / d_1$$

and iterate (k=1,2,...) according to

$$a_{k+1} = \frac{1}{1 + \frac{n_{k+1}}{d_k d_{k+1}} a_k}$$

In the limit, the q sequence converges to the value of the continued fraction

$$b_{k+1} = (a_k - 1)b_k,$$

$$q_{k+1} = q_k + b_{k+1},$$

6.3 Numerical Examples

In the following numerical applications we consider the value of the Hubble constant as $H_0 = 55 \text{ km/sec/Mpc}$

- **For $\sigma_0 = q_0$**

1- $\sigma_0 = q_0 = 0.4, X = 0.02 (t = 35.02 \times 10^7 \text{ years}) \Rightarrow Y = 1.01992105$

2- $\sigma_0 = q_0 = 0.6, X = -0.04 (t = -72.04 \times 10^7 \text{ years}) \Rightarrow Y = 0.9595067$

3- $\sigma_0 = q_0 = 0.5, X = 0.4 (t = 720.4 \times 10^7 \text{ years}) \Rightarrow Y = 1.36797989$

- **For $\sigma_0 \neq q_0$**

4-

$\sigma_0 = 0.5, q_0 = 0.34, X = 0.03 (t = 54.03 \times 10^7 \text{ years}) \Rightarrow Y = 1.02985211$

5-

$\sigma_0 = 1.0, q_0 = 2.0, X = -0.1 (t = -180.1 \times 10^7 \text{ years}) \Rightarrow Y = 0.88979408$

6-

$\sigma_0 = 4.0, q_0 = 5.0, X = 0.1 (t = 180.1 \times 10^7 \text{ years}) \Rightarrow Y = 1.07595569$

In concluding the present paper, the authors hope that the literal analytical solutions in power series forms developed for the radius of curvature and the expansion velocity of the zero temperature cosmological model of the Ideal Universe is most useful in the search for Quantum Gravity^[15]. The importance of these solutions is due to their analytical forms which offer, in general, much deeper insight into the nature of the physical characteristics to which they refer.

On the other hand we proposed for the computational developments of these expressions the continued fraction theory. The continued fraction has not given the prominence they deserve in the university curricula despite the fact that they are, generally, far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series and, ironically, they were in use centuries before the invention of the power series. Finally some numerical examples are also given.

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دراسات على جاذبية الكم أ- نموذج تحليلي للكون المثالي ذي الخلفية منعدمة الحرارة

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المستخلص. في هذا البحث من متسلسلة البحوث، درسنا نموذج الكون المثالي ذي الخلفية منعدمة الحرارة. وقد تم الوصول إلى حل تحليلي لهذا النموذج الكوني على صورة متسلسلة قوى. استغل هذا الحل لإيجاد نصف قطر الانحناء للكون المثالي وتمدد سرعته، وذلك عند أي زمن، وللوصول لهذا الحل فقد استخدمنا طريقة الكسر المستمر. احتوى البحث أيضاً على بعض التطبيقات العددية.