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An Interactive Method for Fuzzy Multiobjective Nonlinear Programming Problems

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Abstract. This paper presents a solution method for multiobjective nonlinear programming problems with fuzzy parameters in the objective functions. This method is based on interactive cutting-plane algorithm. Also, the stability set corresponding α-Pareto optimal solution which is obtained by using this method is investigated. The method is illustrated by a numerical example.

1. Introduction

In an earlier work, Osman[1,2] introduced the notions of the solvability set, stability set of the first kind and the second kind, and analyzed these concepts for parametric convex nonlinear programming problems. Osman and El-Banna[3] introduced the qualitative analysis of the stability set of the first kind for fuzzy parametric multiobjective nonlinear programming problems. Kassem[4] introduced the interactive stability of multiobjective nonlinear programming problems with fuzzy parameters in the constraints. Sakawa and Yano[5] introduced the concept of α-multiobjective nonlinear programming and α-Pareto optimality. Loganathan and Sherali[6] presented an interactive cutting-plane algorithm for determining a best-compromise solution to a multiobjective optimization problem in situations with an implicitly defined utility function. Recently, Elshafei[7] introduced an interactive stability compromise programming method for solving fuzzy multiobjective integer nonlinear programming problems.

This paper presents an interactive cutting-plane algorithm for solving multiobjective nonlinear programming problems with fuzzy parameters in the objective functions. In this algorithm, if the decision maker (DM) is unsatisfied with the corresponding α-Pareto optimal solution with the degree α, the DM updates the degree α of α-level set by considering the stability set which has the same corresponding α-Pareto optimal solution.

2. Problem Formulation

Let us consider the following fuzzy multiobjective nonlinear programming (FMONLP) problem:

$$\text{(FMONLP)} \quad \begin{array}{l} \max\, (f_1(x,\tilde{a}_1), f_2(x,\tilde{a}_2),\ldots, f_k(x,\tilde{a}_k)) \\ \textit{subject to}\; x \in X = \{x \in R^n \mid g_j(x) \le 0,\; j = 1,2,\ldots,m\}, \end{array}$$

where $f_i(x,\tilde{a}_i)$ is a continuously differentiable and concave function for i=1,2,…,k, X is nonempty convex and compact, g_j is a continuously differentiable and convex function for j=1,2,…,m, and $\tilde{a}_i = (\tilde{a}_{i1}, \tilde{a}_{i2},\ldots,\tilde{a}_{iq_i})$ represented a vector of fuzzy parameters in the objective function $f_i(x,\tilde{a}_i)$, these fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced in [8]. A real fuzzy number $\tilde{p}$ is a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{p}}(p)$ is defined as:

(1) a continuous mapping from R to the closed interval [0,1].
(2) $\mu_{\tilde{p}}(p) = 0$ for all $p \in (-\infty, p_1]$.
(3) strict increase on (p_1, p_2).
(4) $\mu_{\tilde{p}}(p) = 1$ for all $p \in [p_2, p_3]$.
(5) strict decrease on (p_3, p_4).
(6) $\mu_{\tilde{p}}(p) = 0$ for all $p \in [p_4, +\infty)$.

For simplicity of notations we define the following vectors:

$$\tilde{a}_i = (\tilde{a}_{i1}, \tilde{a}_{i2},\ldots,\tilde{a}_{iq_i})$$
$$a = (a_1, a_2,\ldots, a_k)$$
$$\tilde{a} = (\tilde{a}_1, \tilde{a}_2,\ldots,\tilde{a}_k)$$

Here, we assume that the membership function $\mu_{\tilde{a}}(a)$ is differentiable on $[p_1, p_4]$ and the problem (FMONLP) is stable[9].

Definition 1 (*α-level set*). The α-level set of the numbers $\tilde{a}_i$ (i=1, 2, …, k) is defined as the ordinary set $L_\alpha(\tilde{a})$ for which the degree of their membership functions exceeds the level α:

$$L_\alpha(\tilde{a}) = \{a \mid \mu_{\tilde{a}_i}(a_i) \geq \alpha, \quad i = 1,2,...,k\}.$$

For a certain degree of α, the problem (FMONLP) can be understood as the following nonfuzzy α-multiobjective nonlinear programming (α-MONLP)' problem:

$$(\alpha\text{-MONLP})' \quad \begin{array}{l} \max\,(f_1(x,a_1), f_2(x,a_2),..., f_k(x,a_k)) \\ \textit{subject to } x \in X, a \in L_\alpha(\tilde{a}). \end{array}$$

Problem (α-MONLP)' can be rewritten as the following form:

$$(\alpha\text{-MONLP}) \quad \begin{array}{l} \max\,(f_1(x,a_1), f_2(x,a_2),..., f_k(x,a_k)) \\ \textit{subject to } x \in X, \\ \qquad A_i \leq a_i \leq B_i, i = 1,2,...,k, \end{array}$$

where A_i, B_i are lower and upper bounds on a_i for i=1,2,…,k. Since (FMONLP) is stable, the problem (α-MONLP) is also stable.

Definition 2 (*α-Pareto optimal solution*). $x^* \in X$ is said to be an α-Pareto optimal solution to the (α-MONLP), if and only if there does not exist another $x \in X$, $a \in L_\alpha(\tilde{a})$ such that $f_i(x,a_i) \geq f_i(x^*,a_i^*)$, i=1,2,…,k, with strictly inequality holding for at least one i, where the corresponding values of parameters a_i^* are called α-level optimal parameters.

For some (unknown) implicit utility function we have the following problem:

$$(\alpha\text{M}) \quad \begin{array}{l} \max\; U[f_1(x,a_1), f_2(x,a_2),..., f_k(x,a_k)] \\ \textit{subject to } x \in X, \\ \qquad A_i \leq a_i \leq B_i, i = 1,2,...,k, \end{array}$$

where $U(.)$ is concave and continuously differentiable. It is clear that (x^*, a^*) is an α-Pareto optimal solution of (α-MONLP) if and only if (x^*, a^*) is optimal solution of (αM).

3. Stability Set of the First Kind

Definition 3 (*stability set of the first kind*). Suppose that $(\overline{A}, \overline{B}) \in R^{2k}$ with a corresponding α-Pareto optimal solution $(\overline{x}, \overline{a})$, then the stability set of the first kind of α-MONLP corresponding to $(\overline{x}, \overline{a})$, denoted by $S(\overline{x}, \overline{a})$, is defined by

$$S(\overline{x}, \overline{a}) = \{(A, B) \in R^{2k} \mid (\overline{x}, \overline{a}) \text{ is an } \alpha \text{ - Pareto optiomal solution of } (\alpha \text{ - MONLP})\}.$$

Let a certain $(\overline{A}, \overline{B})$ with a corresponding α-Pareto optimal solution $(\overline{x}, \overline{a})$ be given, then from the stability of problem (α-MONLP) there exist $(A, B) \in R^{2k}, \nu \in R^m$, $\lambda \in R^k$, $\delta \in R^k$, and $\mu \in R^k$ such that the following Kuhn-Tucker conditions are satisfied:

$$\sum_{i=1}^{k} \lambda_i \frac{\partial f_i}{\partial x_r}(\overline{x}, \overline{a}) - \sum_{j=1}^{m} \nu_j \frac{\partial g_j}{\partial x_r}(\overline{x}) = 0, r = 1,2,\ldots,n \qquad (1)$$

$$\lambda_i \frac{\partial f_i}{\partial a_i}(\overline{x}, \overline{a}) - \delta_i + \mu_i = 0, \qquad i = 1,2,\ldots,k \qquad (2)$$

$$\sum_{i=1}^{k} \lambda_i = 1, \qquad (3)$$

$$g_j(\overline{x}) \le 0, \qquad j = 1,2,\ldots,m \qquad (4)$$

$$\overline{a}_i - B_i \le 0, \qquad i = 1,2,\ldots,k \qquad (5)$$

$$A_i - \overline{a}_i \le 0, \qquad i = 1,2,\ldots,k \qquad (6)$$

$$\nu_j g_j(\overline{x}) = 0, \qquad j = 1,2,\ldots,m \qquad (7)$$

$$\delta_i(\overline{a}_i - B_i) = 0 \qquad i = 1,2,\ldots,k \qquad (8)$$

$$\mu_i(A_i - \overline{a}_i) = 0 \qquad i = 1,2,\ldots,k \qquad (9)$$

$$\lambda_i, \mu_i, \delta_i, \nu_j \ge 0, \qquad i = 1,2,\ldots,k; \; j = 1,2,\ldots,m \qquad (10)$$

The determination of the stability set of the first kind $S(\overline{x},\overline{a})$ depends only on whether any of the variables δ_i, i=1,2,…,k and any of the variables μ_i, i=1,2,…,k which solve the equations (2) and (10), are positive or zero.

Let $\mu_i = 0$, $i \in I_1 \subseteq \{1,2,...,k\}$; $\mu_i > 0$, $i \notin I_1$, solves (2) and (10), then in order that the other Kuhn-Tucker conditions (6) and (9) are satisfied, we must have

$$A_i \le \overline{a}_i, i \in I_1; A_i = \overline{a}_i, i \notin I_1.$$

Let $\lambda_i = 0$, $i \in I_2 \subseteq \{1,2,...,k\}$; $\lambda_i > 0$, $i \notin I_2$, solves (2) and (10), then in order that the other Kuhn-Tucker conditions (5) and (8) are satisfied, we must have

$$B_i \ge \overline{a}_i, i \in I_2; B_i = \overline{a}_i, i \notin I_2.$$

Let

$$S_{I_1,I_2}(\overline{x},\overline{a}) = \{(A,B) \in R^{2k} \mid A_i \le \overline{a}_i, i \in I_1; A_i = \overline{a}_i, i \notin I_1; \quad B_i \ge \overline{a}_i, i \in I_2; B_i = \overline{a}_i, i \notin I_2\}.$$

Then, it is clear that

$$S(\overline{x},\overline{a}) = \bigcup_{\substack{Possible \\ I_1, I_2}} S_{I_1,I_2}(\overline{x},\overline{a}).$$

4. Interactive Algorithm

In this approach the DM is asked to estimate the precise local tradeoff ratios, these ratios at a point $(x^l, a^l) \in X \times L_\alpha(\widetilde{a})$ are

$$t_i^l = \left. \frac{\partial U(z)/\partial z_i}{\partial U(z)/\partial z_1} \right|_{z=z^l}, \; i=1,2,\ldots,k,$$

where $z = (z_1, z_2, ..., z_k)$, $z_i \equiv f_i(x, a_i)$ for i=1,2,…,k, and

$z^l \equiv (f_1(x^l, a_1^l), ..., f_k(x^l, a_k^l))$. Corresponding to each iterate z^l, the method generates a so-called tradeoff cut. This cut at a point (x^l, a^l) is

$$\sum_{i=1}^{k} t_i^l (z_i - z_i^l) \ge 0, \text{ where } z_i \equiv f_i(x, a_i) \text{ for } i=1,2,\ldots,k.$$

This method attempts to locate the next solution by solving the following problem

$$\begin{aligned} &\max \eta \\ (\alpha \mathrm{P}^{l}) \quad & \textit{subject to}\ \eta \le \sum_{i=1}^{k} t_i^h (z_i - z_i^h)\ , h = 0,1,2,\ldots,l, \\ & (x,a) \in X \times L_\alpha(\widetilde{a}). \end{aligned}$$

The following lemma establishes an important characteristic of problem $\alpha \mathrm{P}^l$.

Lemma: Let $(\overline{x},\overline{a},\overline{\eta})$ solve problem $\alpha \mathrm{P}^l$, and let $\overline{z} \equiv (f_1(\overline{x},\overline{a}_1),\ldots,f_k(\overline{x},\overline{a}_k))$. Then $(\overline{x},\overline{a})$ is an α-Pareto optimal solution to the α-MONLP problem.

Proof: Suppose that $(\overline{x},\overline{a})$ is not α-Pareto optimal solution, then there exists $(x^*,a^*) \in X \times L_\alpha(\widetilde{a})$ such that $z^* \ge \overline{z}$ and $z^* \ne \overline{z}$, where $z^* \equiv (f_1(x^*,a_1^*),\ldots,f_k(x^*,a_k^*))$, then

$\overline{\eta} = \min_{h=0,1,\ldots,l} \{t^h(\overline{z} - z^h)\} < \min_{h=0,1,\ldots,l} \{t^h(z^* - z^h)\}$, which contradicts the optimality of $(\overline{x},\overline{a},\overline{\eta})$. This completes the proof.

The steps of the algorithm can be summarized as follows:

Step 1. Ask the DM to select the initial value of α ($0 \le \alpha \le 1$).

Step 2. Determine the α-level set of the fuzzy numbers.

Step 3. Convert the FMONLP in the form of α-MONLP and select an initial feasible point (x^l, a^l). Set l=0 and let

$$z^l \equiv (f_1(x^l,a_1^l),\ldots,f_k(x^l,a_k^l)).$$

Step 4. The DM specify precise values of t_i^l at (x^l,a^l) for i=1,2, …,k.

Step 5. Solve problem $\alpha \mathrm{P}^l$. Let $(x^{l+1},a^{l+1},\eta^{l+1})$ be an optimal solution, and let $z^{l+1} \equiv (f_1(x^{l+1},a_1^{l+1}),\ldots,f_k(x^{l+1},a_k^{l+1}))$. Set l=l+1.

Step 6. If $\eta^l = 0$ go to Step 7. Otherwise, go bake to Step 4.

Step 7. Determine the stability set of the first kind $S(\overline{x},\overline{a})$.

Step 8. If the DM is satisfied with the current values of the objective functions and α of α-Pareto optimal solution, go to Step 9. Otherwise, ask the DM to update the degree α and return to Step 2.

Step 9. Terminate with (x^l, a^l) as the final solution.

5. Illustrative Numerical Example

Let us consider the following fuzzy multiobjective nonlinear programming problem

(FMONLP)

$\max(x_1 + \tilde{a}_1, x_2 + \tilde{a}_2)$

subject to $x \in X = \{(x_1, x_2) \mid -x_1 + x_2 \le 3, x_1^2 + x_2^2 \le 25, x_1, x_2 \ge 0\}$,

$$\text{with } \mu_{\tilde{a}_i}(a_i) = \begin{cases} 0 & -\infty < a_i \le p_1 \\ \dfrac{a_i - p_1}{p_2 - p_1} & p_1 < a_i < p_2 \\ 1 & p_2 \le a_i \le p_3 \\ \dfrac{a_i - p_4}{p_3 - p_4} & p_3 < a_i \le p_4 \\ 0 & p_4 \le a_i < \infty \end{cases}$$

where i=1,2 and the values of p_j (j=1,2,3,4) are given in the following table:

$\tilde{a}_i$	p_1	p_2	p_3	p_4
$\tilde{a}_1$	3.8	4	4.8	5
$\tilde{a}_2$	1	2	3	4

Let $z_1 = f_1(x, a_1) = x_1 + a_1$, $z_2 = f_2(x, a_2) = x_2 + a_2$. The DM's utility function U is assumed to be the following:

$$U = -(z_1 - 20)^2 - 2(z_2 - 10)^2$$

Step 1: Suppose that the DM select α=0.9.

Step 2: $L_{0.9}(\tilde{a}) = \{(a_1, a_2) \mid 3.98 \le a_1 \le 4.82 \text{ and } 1.9 \le a_2 \le 3.1\}$

Step 3: The α-MONLP problem becomes:

$$\max\,(x_1 + a_1, x_2 + a_2)$$
$$subject\ to\ x \in X, a \in L_{0.9}(\tilde{a})$$

Let us select the feasible point (x^0, a^0)=(3.18, 2.9, 4.82, 3.1) as the starting solution. Hence, the corresponding point in the objective space is $z^0 \equiv (8,6)$.

Step 4: The local tradeoff ratio t^0 at (x^0, a^0) is t^0=(1, 0.666667).

Step 5: Solve the following problem:

$$\max\ \eta$$
$$(\alpha \mathrm{P}^0)\quad subject\ to\ \eta \le x_1 + 0.666667x_2 + a_1 + 0.666667a_2 - 12.000002$$
$$x \in X, a \in L_{0.9}(\tilde{a})$$

The solution to this problem is

(x^1, a^1, η^1)=(4.160220,2.773566,4.82,3.1,0.895931), and

$$z^1 \equiv (8.980220, 5.873566).$$

Step 6: Since $\eta^1 \neq 0$ then return to Step 4 for the second iteration.

2^{nd} *Iteration* (from Step 4 to Step 6)

We have (x^2, a^2)=(4.002186,2.997098,4.82,3.1)
with η^2=0.009372, $z^2 \equiv$(8.822186,6.097098).

3^{rd} *Iteration* (from Step 4 to Step 6)

We get (x^3, a^3)=(4.098459,2.864042,4.82,3.1)
with η^3=0.003356, $z^3 \equiv$(8.918459,5.964042).

4^{th} *Iteration* (from Step 4 to Step 6)

We obtain (x^4, a^4)=(4.041529,2.943816,4.82,3.1)
with η^4=0.001178, $z^4 \equiv$(8.861529,6.043816).

5^{th} *Iteration* (from Step 4 to Step 6)

We have (x^5, a^5)=(4.078205,2.892808,4.82,3.1)
with η^5=0.000443, $z^5 \equiv$(8.898205,5.992808).

6th Iteration (from Step 4 to Step 6)

We have (x^6, a^6)=(4.054285,2.926233,4.82,3.1)

with η^6=0.000209, $z^6 \equiv$(8.874285,6.026233).

7th Iteration (from Step 4 to Step 6)

We have (x^7, a^7)=(4.065676,2.910374,4.82,3.1)

with η^7=0.000063, $z^7 \equiv$(8.885676,6.010374).

8th Iteration (from Step 4 to Step 6)

We have (x^8, a^8)=(4.062271,2.915142,4.82,3.1)

with $\eta^8 \approx 0$, $z^8 \equiv$(8.882271,6.015142).

Step 7: The corresponding stability set of the first kind is:

$$S(x^8, a^8) = \{(A, B) \mid A_1 \le 4.82, B_1 = 4.82, A_2 \le 3.1, B_2 = 3.1\}.$$

Step 8: Suppose that the DM is satisfied with the z^8 and α=0.9 . Go to Step 9.

Step 9: The final solution is (x^8, a^8)=(4.062271,2.915142,4.82,3.1).

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الطريقة التفاعلية لمسائل البرمجة غير الخطية المتعددة الأهداف الضبابية

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المستخلص. يقدم هذا البحث طريقة لحل مسائل البرمجة غير الخطية المتعددة الأهداف مع وجود وسائط ضبابية في دوال الهدف، وتعتمد تلك الطريقة على خوارزمية المستوى القاطع التفاعلية.

كما تم التحقق في مجموعة الاستقرار المرافقة لحل α− بارتو الأمثل الذي حصلنا علية باستخدام تلك الطريقة. والطريقة المقدمة موضحة بمثال عددي.