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Some Inequalities on Statistical ($\overline{N}$, *p*)-Summability

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Abstract. We prove some inequalities related to the concepts of $\overline{N}$ (*st*)-conservative matrices, $\overline{N}$ (*st*)-lim sup and $\overline{N}$ (*st*)-lim inf which are natural analogues of (c, $st \cap l_\infty$)-matrices, *st*-lim sup and *st*-lim inf respectively.

1. Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences of real numbers with the usual supremum norm. Let $A = (a_{nk})$, n, $k \in$ N, be an infinite matrix of real numbers, and let $x = (x_k)$ be a sequence of real numbers. We write $Ax = (A_n(x))$ if $A_n = \sum_k a_{nk}x_k$ converges for each n. Let X and Y be any two sequence spaces. If $x \in X$ implies $Ax \in Y$, then we say that the matrix A maps X into Y. We denote the class of all matrices A which map X into Y by (X,Y). If X and Y are equipped with X-lim and Y-lim, A $\in (X,Y)$ and Y-lim $Ax = X$-lim x for all $x \in X$, then we write $A \in (X,Y)_{reg}$.

Let $E \subseteq$ N. Natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} | \{k \le n : k \in E\} |,$$

where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be *statistically convergent*[1] to L, st - lim x = L, if for every $\epsilon > 0$, $\delta(\{k : |x_k - L| \ge \epsilon\}) = 0$. Statistical convergence for double sequences has been defined and studied by Mursaleen and Edely[2].

It is known that A (c; c), that is, A is conservative if and only if

$$(\mathrm{i})\ \|A\| = \sup_n \sum_k |a_{nk}| < \infty,$$

$$(\mathrm{ii})\ a_k = \lim_n a_{nk}, \text{ for each } k,$$

$$(\mathrm{iii})\ a = \lim_n \sum_k a_{nk}$$

If A is conservative, the number $\chi = \chi(A) = a \sum_k ak$ is called the characteristic of A. A is said to be *regular* if and only if (i), (ii) with $a_k = 0$ for all k; and (iii) with $a = 1$ hold.

Also, It is known that A is *st-conservative*, *i.e. A* $\in$ (c, $(st) \cap \ell_\infty$), if and only if

$$(\mathrm{i})\ \|A\| = \sup_n \sum_k |a_{nk}| < \infty,$$

$$(\mathrm{ii})\ t_k = st - \lim_n a_{nk} \text{ for each } k,$$

$$(\mathrm{iii})\ t = st - \lim_n \sum_k a_{nk}$$

Here $\chi_{st} = \chi_{st}(A) = t - \sum_k t_k$ is called the statistical *characteristic* of A.

Recently, Moricz and Orhan[3] defined the concept of statistical $(\overline{N}, p)$-summability as follows. Let $p = (p_k)_{k=0}^{\infty}$ be a sequence of nonnegative numbers such that $p_0 > 0$ and $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$, and write $t_n = P_n^{-1} \sum_{k=0}^{n} p_k x_k$. We say that $x = (x_k)$ is *statistically* $(\overline{N}, p)$-summable to L if the sequence $t = (t_n)$ is statistically convergent to L, *i.e.* st-lim $t = L = \overline{N}$ (st)-lim x. We denote by $\overline{N}$ (st) the set of all sequences which are statistically $(\overline{N}, p)$-summable.

In this paper we prove some inequalities related to the concepts of $\overline{N}$ (st)-conservative matrices, $\overline{N}$ (st)-lim sup and $\overline{N}$ (st)-lim inf which are natural analogues of (c, $st \cap \ell_\infty$)-matrices, st-lim sup and st-lim inf respectively (see

Kolk[4] and Fridy and Orhan[5]). Such type of inequalities are also considered by Coskun and Cakan[6], and Cakan and Altay[7].

2. Needed Lemmas

Lemma 2.1 *[Das[8], Theorem 1(c)]*

Let A be conservative. Then, for some constant $\lambda \geq |\chi|$ and for all $x \in \ell_\infty$,

$$\lim_n \sup \sum \left(a_{nk} - a_k \right) x_k \leq \frac{\lambda + X}{2} L(x) - \frac{\lambda - X}{2} l(x)$$

if and only if

$$\lim_n \sup \sum |a_{nk} - a_k| \leq \lambda.$$

Here, L(x) = lim sup x and l(x) = lim inf x.

Lemma 2.2 *[Simons[9], Corollary 12]*

Let $\|A\| < \infty$ and $\lim_n |a_{nk}| = 0$. *Then there exists a* $y \in \ell_\infty$ *such that* $\|y\| \leq 1$ *and*

$$\lim_n \sup \sum_k |a_{nk}| = \lim_n \sup \sum_k a_{nk} y_k$$

Lemma 2.3 *[Das[8], Lemma 1]*

Suppose that A is conservative and $\lambda \geq 0$ *Then*

$$\lim_n \sup \sum_k |a_{nk} - a_k| \leq \lambda$$

if and only if

$$\lim_n \sup \sum_{k \notin E} \left(a_{nk} - a_k \right)^+ x_k \leq \frac{\lambda + \chi}{2} \beta(x)$$

and

$$\lim_n \sup \sum_{k \notin E} \left(a_{nk} - a_k \right)^- x_k \leq \frac{\lambda - \chi}{2} \alpha(x)$$

where E is any subset of N with $\delta(E) = 0$.

Lemma 2.4 *$A \in (c, (st) \cap \ell_\infty)$ if and only if*

(i) $||A|| = sup_n \sum_k | a_{nk} | < \infty$,

(ii) N(st) - $lim_n\, a_{nk} = \alpha_k$ *for every k; and*

(iii) N(st) - $lim_n \sum_k a_{nk} = \alpha$.

We call such matrices as N(st)-*conservative* matrices, and in this case $k = \alpha - \sum_k \alpha_k$ is defined which is known as the N(st)-*characteristic* of A.

3. Main Results

Theorem 3.1 *Let A be conservative and $x \in \ell_\infty$ .Then*

$$\lim_n \sup \sum_k (a_{nk} - a_k) x_k \leq \frac{\lambda + \chi}{2} \beta(x) - \frac{\lambda - \chi}{2} \alpha(x) \tag{1}$$

for some constant $\lambda \geq |\chi|$, if and only if

$$\lim_n \sup \sum_k |a_{nk} - a_k| \leq \lambda, \tag{2}$$

$$\lim_n \sum_{x \in E} |a_{nk} - a_k| = 0 \tag{3}$$

*for every $E \subseteq$ **N** with δ (E) = 0; where β (x) = N(st) - lim sup x and α (x) = N(st) - lim inf x.*

Proof: *Necessity*. Let $L(x)$ = lim sup x and $l(x)$ = lim inf x: Since β(x) $\leq L(x)$ and $l(x) \leq \alpha$(x) which gives $-\alpha(x) \leq -l(x)$ for all $x \in \ell_\infty$, we have using (1)

$$\lim_n \sup \sum_k (a_{nk} - a_k) x_k \leq \frac{\lambda - \chi}{2} L(x) - \frac{\lambda - \chi}{2} l(x).$$

Using Lemma 2.1, we see that the necessity of (2) is proved. Now to prove the necessity of (3) let us define the matrix B = (b_{nk}) by

$$b_{nk} = \begin{cases} a_{nk} - a_k, \text{for } k \in E; \\ 0, \text{otherwise,} \end{cases}$$

where E is any subset of **N** with δ (E) = 0. Since A is conservative, the matrix B satisfies the conditions of Lemma 2.2. Thus there exists a $y \in \ell_\infty$ such that $\|y\| \leq$ 1 and

$$\limsup_n \sum_k |b_{nk}| = \limsup_n \sum_k b_{nk} y_k. \qquad (4)$$

Now, let $y = (y_k)$ be defined as

$$y_k = \begin{cases} 1, \text{ for } k \in E, \\ 0, \text{ for } k \notin E. \end{cases}$$

Hence N(st) - lim y = $\beta(y) = \alpha(y) = 0$. Using (1) and (4) we have

$$\limsup_n \sum_{k \in E} |a_{nk} - a_k| \leq \frac{\lambda + X}{2} \beta(y) - \frac{\lambda - X}{2} \alpha(y) = 0$$

and this proved the necessity of (3).

Sufficiency. Let $x \in \ell_\infty$. Now let us define the following sets as

$$E_1 = \{k : t_k > \beta(x) + \epsilon\} \text{ and } E_2 = \{k : t_k < \alpha(x) - \epsilon\}$$

It is clear to see that $\delta(E_1) = \delta(E_2) = 0$; and hence $\delta(E) = 0$ for $E = E_1 \cap E_2$. The following can be written as

$$\sum_k (a_{nk} - a_k) x_k = \sum_{k \in E} (a_{nk} - a_k) x_k + \sum_{k \notin E} (a_{nk} - a_k) x_k$$

$$= \sum_{k \in E} (a_{nk} - a_k) x_k + \sum_{k \notin E} (a_{nk} - a_k)^+ x_k - \sum_{k \notin E} (a_{nk} - a_k)^- x_k.$$

Using (3), we have lim sup$_n$ $\sum_{k \in E} (a_{nk} - a_k) x_k = 0$. Using Lemma 2.3, we have

$$\limsup_n \sum_{k \notin E} (a_{nk} - a_k)^+ x_k \leq \frac{\lambda + \chi}{2} \beta(x)$$

and

$$\limsup_n \sum_{k \notin E} (a_{nk} - a_k)^- x_k \leq \frac{\lambda - \chi}{2} \alpha(x)$$

Thus, using (5),

$$\mathit{lim}\ \underset{n}{\mathit{sup}} \sum_k (a_{nk} - a_k) x_k \le \frac{\lambda + \chi}{2} \beta(x) - \frac{\lambda - \chi}{2} \alpha(x)$$

which completes the proof of sufficiency.

To prove next theorems, we need the following lemma which is $N(st)$ analogue of Lemma 2.2.

Lemma 3.1 Let $\|A\| < \infty$ and $N(st) - \lim_n |a_{nk}| = 0$. *Then there exists a* $y \in \ell_\infty$ *such that* $\|y\| \le 1$ *and*

$$N(st) - \lim\ \sup \sum_{k} a_{nk} y_k = N(st) - \lim \sup \sum_k |ank|.$$

The following lemma is derived by replacing *st* with $N(st)$ from Lemma 2.3 of [Coskun and Cakan[6]].

Lemma 3.2 *Let A be st-conservative and* $\lambda > 0$. *Then*

if and only if
$$N(st) - \lim\ \underset{n}{\sup} \sum_{k} |a_{nk} - \alpha_k| \le \lambda$$

and
$$N(st) - \lim\ \underset{n}{\sup} \sum_{k} (a_{nk} - a_k) + \le \frac{\lambda + \chi st}{2}$$
$$N(st) - \lim\ \underset{n}{\sup} \sum_{k} (a_{nk} - a_k) - \le \frac{\lambda + \chi st}{2}.$$

Theorem 3.2 *Let A be N(st)-conservative. Then, for some constant* $\lambda \ge |k|$ *and for all* $x \in \ell_\infty$

$$N(st) - \lim\ \underset{n}{\sup} \sum_{k} (a_{nk} - \alpha_k) x_k \le \frac{\lambda + k}{2} L(x) - \frac{\lambda - k}{2} l(x)$$

if and only if

$$N(st) - \lim\ \underset{n}{\sup} \sum_{k} |ank - \alpha k| \le \lambda$$

Proof: <u>Necessity</u>. Let us define the matrix $B = (b_{nk})$ by $b_{nk} = a_{nk} - \alpha_k$ for all *n, k*. Since *A* is $N(st)$-conservative, we have $N(st)\text{-}\lim_n a_{nk} = \alpha_k$ and hence

$$N(st) - \lim_n |b_{nk}| = N(st) - \lim_n |a_{nk} - a_k| = 0$$

Thus the matrix B satisfies the hypothesis of Lemma 3.1. Hence there exists a $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$N(st) - \lim\ \sup \sum_{k} |bnk| = N(st) - \lim \sup \sum_{k} bnkyk.$$

Thus, using (6),

$$\begin{aligned} N(st) - \lim_{n}\ \sup \sum_{k} |bnk| &= N(st) - \lim_{n}\ \sup \sum_{k} bnkyk \\ &= N(st) - \lim_{n}\ \sup \sum_{k} (ank - \alpha\, k) yk \\ &\leq \frac{\lambda + k}{2} L(y) - \frac{\lambda - k}{2} l(y) \\ &\leq \left(\frac{\lambda + k}{2} + \frac{\lambda - k}{2} \right) \|y\| \\ &\leq \lambda, \end{aligned}$$

which completes the proof of (7).

<u>Sufficiency</u>. Let (7) holds and $x \in \ell_\infty$. Now, for some $k_0 \in \mathbf{N}$ $(k > k_0)$; we can write

$$\sum_{k} (a_{nk} - a_k) xk = \sum_{k \leq ko} (a_{nk} - a_k) x_k + \sum_{k \succ ko} (a_{nk} - a_k) + x_k - \sum_{k \succ ko} (a_{nk} - a_k) - x_k.$$

Since for any $\in > 0$, $l(x) - \in < x_k < L(x) + \in$; and A is $N(st)$-conservative, and using Lemma 2.3, we get

$$\begin{aligned} N(st) - \lim_{n}\ \sup \sum_{k} (ank - ak) xk &\leq (L(x) + E) \left(\frac{\lambda + k}{2} \right) - (l(x) - E) \left(\frac{\lambda - k}{2} \right) \\ &= \frac{\lambda + k}{2} L(x) - \frac{\lambda - k}{2} l(x) + \lambda E, \end{aligned}$$

which give (6), since $\in$ was arbitrary.

Theorem 3.3 *Let A be N(st)-conservative. Then, for some constant* $\lambda \geq |k|$ *and for all* $x \in \ell_\infty$

$$N(st) - \lim_{n}\ \sup \sum_{k} (ank - \alpha\, k) \leq \frac{\lambda + k}{2} \beta(x) - \frac{\lambda - k}{2} \alpha(x) \tag{8}$$

if and only if (7) hold and

$$N(st) - \lim_{n} \sum_{k \in E} (ank - \alpha k) = 0 \tag{9}$$

for every E $\in$ **N** *with* δ(E) = 0.

Proof: *Necessity*. Let (8) hold. Since $\beta(x) \leq L(x)$ and $-\alpha(x) \leq l(x)$, the necessity of (7) follows from Theorem 3.2. Now let us show the necessity of (9). For any E $\subseteq$ **N** with $\delta(E) = 0$; let us define a matrix $B = (b_{nk})$ by the following

$$b_{nk} = \begin{cases} a_{nk} - a_k, & \text{for } k \in E, \\ 0, & \text{otherwise} \end{cases}$$

Then, it is clear that B satisfies the condition of Lemma 3.1 and hence there exists a $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$N(st) - \lim \sup_n \sum_{\text{k}} b_{nk} y_k = N(st) - \lim \sup_n \sum_{\text{k}} |\text{b}_{\text{nk}}|$$

Let us define the sequence $y = (y_k)$ by

$$yk = \begin{cases} 1, \text{ for } k \in E, \\ 0, \text{ for } k \notin E. \end{cases}$$

Using the fact that $N(st)$ - lim $y = \beta(y) = \alpha(y) = 0$; and using (8) and (10), we get

$$\begin{aligned} N(st) - \lim \sup_n \sum_{\text{k}} |a_{nk} - \alpha_k| &= N(st) - \lim \sup_n \sum_{\text{k}} |b_{nk}| \\ &= N(st) - \lim \sup_n \sum_{\text{k}} b_{nk} y_k \\ &\leq \frac{\lambda + k}{2} \beta(y) - \frac{\lambda - k}{2} \alpha(y) = 0, \end{aligned}$$

and hence we get the necessity of (9).

Sufficiency. Let (7) and (9) hold and $x \in \ell_\infty$. Put the set E as in Theorem 3.1. Now we can write

$$\sum_k (a_{nk} - \alpha_k) x_k = \sum_{k \in E} (a_{nk} - \alpha_k) x_k + \sum_{k \notin E} (a_{nk} - \alpha_k) + x_k - \sum_{k \notin E} (a_{nk} - \alpha_k) - x_k.$$

Using Lemma 2.3 and (9), we have

$$N(st) - \lim \sup \sum_{\text{k}} (a_{nk} - \alpha_k) \leq \frac{\lambda + k}{2} \beta(x) - \frac{\lambda - k}{2} \alpha(x) + \lambda E.$$

But $\in$ was arbitrary, so (8) holds and this completes the proof of Theorem 3.3.

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إثبات بعض المتراجحات المتعلقة بمفهوم مصفوفات $(\bar{N}, p)$

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المستخلص. في هذه الورقة العلمية تم إثبات بعض المتراجحات والمتعلقة بمفهوم مصفوفات $N(st)$ ، ونهاية $N(st)$ العليا، ونهاية $N(st)$ ، والتي تعتبر تعميما لمفهوم مصفوفات st ، ونهاية st العليا، ونهاية st.