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Derivations of *B*-algebras

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Abstract. The notion of left-right (resp. right-left) derivation of *B*-algebra is introduced and some related properties are investigated. Also the notion of derivation of 0-commutative *B*-algebra is studied and some of its properties are investigated.

1. Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: *BCK*- algebras and *BCI*-algebras[1,2]. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI* -algebras. In[3,4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: *BCH* - algebras. They have shown that the class of *BCI*- algebras is a proper subclass of the class of *BCH*- algebras. In[5] the authors introduced the notion of *d*-algebras, which is another useful generalization of *BCK*- algebras, and then they investi-gated several relations between *d*-algebras and *BCK*- algebras as well as some other interesting relations between *d*-algebras and oriented digraphs. Y.B. Jun, E. H. Roh and H. S. Kim[6] introduce a new notion, called *BH*-algebras, which is a generalization of *BCH BCI BCK*-algebras. They also defined the notions of ideals in *BH*-algebras. Recently J. Neggers and H. S. Kim[7] introduced the notion of *B*-algebra, and studied some of its properties. In[8] Y. B. Jun and Xin applied the notions of derivations in rings and near rings theory to *BCI*-algebras, and obtained some related properties. In this paper, we apply the notion of left-right (resp. right–left) derivation in *BCI*-algebras to *B*- algebra and investigate some of its properties. Using the concept of derivation of 0-commutative *B*-algebra we investigate some of its properties.

2. Preliminaries

Definition 2.1.[7] A B-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x*x=0$,

(II) $x*0=x$,

(III) $(x*y)*z=x*(z*(0*y))$,

for all $x, y, z \in X$.

Proposition 2.2.[7] If $(X,*,0)$ is a B-algebra then:

(1) $(x*y)*(0*y)=x$,

(2) $x*(y*z)=(x*(0*z))*y$,

(3) $x*y=0$ *implies* $x=y$,

(4) $0*(0*x)=x$.

for all $x, y, z \in X$.

Theorem 2.3.[7] $(X,*,0)$ is a B-algebra if and only if it satisfies the following axioms:

(5) $x*x=0$,

(6) $0*(0*x)=x$,

(7) $(x*z)*(y*z)=x*y$,

(8) $0*(x*y)=y*x$.

for all $x, y, z \in X$.

Theorem 2.4[9] In any B-algebra, the left and right cancellation laws hold.

Definition 2.5.[10] A B-algebra $(X,*,0)$ is said to be 0-commutative if:

$$x*(0*y)=y*(0*x)$$

for all $x, y \in X$

Proposition 2.6.[10] If $(X, *, 0)$ is a 0-commutative *B*-algebra, then

$(9)\ (0*x)*(0*y) = y*x$.

$(10)\ (z*y)*(z*x) = x*y$.

$(11)\ (x*y)*z = (x*z)*y$.

$(12)\ [x*(x*y)]*y = 0$.

$(13)\ (x*z)*(y*t) = (t*z)*(y*x)$.

for all $x, y, z, t \in X$.

From (12) and (3) we get that, If $(X, *, 0)$ is a 0-commutative *B*-algebra, then:

$(14)\ \ x*(x*y) = y$

for all $x, y \in X$.

Definition 2.7.[2] Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a *BCI*-algebra if it satisfies the following axioms:

BCI-1 $((x*y)*(x*z))*(z*y) = 0$;

BCI-2 $(x*(x*y))*y = 0$;

BCI-3 $x*x = 0$;

BCI-4 $x*y = 0$ *and* $y*x = 0$ *imply* $x = y$.

for all $x, y, z \in X$.

For a *BCI*-algebra X, we denote $x \wedge y = y*(y*x)$ for all $x, y \in X$.

Definition 2.8.[8] Let X be a BCI-algebra. By a (ℓ, r)-derivation of X. We mean a self–map d of X satisfying the identity $d(x*y) = (d(x)*y) \wedge (x*d(y))$ for all $x, y \in X$. If X satisfies the identity $d(x*y) = (x*d(y)) \wedge (d(x)*y)$ for all $x, y \in X$, then we say that d is a (r, ℓ)-derivation of X. Moreover if d is both a (ℓ, r)- and a (r, ℓ)-derivation, we say that d is a derivation.

Definition 2.9.[8]. A self map d of a $BCI-$ algebra X is said to be regular if $d(0)=0$.

3. Derivations of *B*-algebra

For a *B*-algebra X, we denote $x \wedge y = y*(y*x)$ for all $x, y \in X$.

Definition 3.1. Let X be a *B*-algebra. By a (ℓ, r)-derivation of X, we mean a self-map d of X satisfying the identity $d(x*y) = (d(x)* y) \wedge (x*d(y))$, for all $x, y \in X$. If X satisfies the identity $d(x*y) = (x*d(y)) \wedge (d(x)* y)$, for all $x, y \in X$, then we say that d is a (r, ℓ)- derivation of X. Moreover, if d is both a (ℓ, r) and a (r, ℓ)- derivation, we say that d is a derivation of X.

Example 3.2. Let $X = \{0,1,2,3\}$ be a *B*-algebra with Cayley table (Table1) as follows:

Table 1

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Define a map $d: X \to X$ by :

$$d(x) = \begin{cases} 3 & if \quad x = 0, \\ 2 & if \quad x = 1, \\ 1 & if \quad x = 2, \\ 0 & if \quad x = 3. \end{cases}$$

Then it is easily checked that d is a derivation of X.

Example 3.3. Let $\mathbb{Z}$ be the set of all integers "–" the minus operation on $\mathbb{Z}$. Then $(\mathbb{Z},-,0)$ is a *B*-algebra. Let $d(x)=x-1$ for all $x\in\mathbb{Z}$.Then

$$\begin{aligned}\left(d(x)-y\right)\wedge\left(x-d(y)\right)&=\left((x-1)-y\right)\wedge\left(x-(y-1)\right)\\&=\left(x-y-1\right)\wedge\left(x-y+1\right)\\&=\left(x-y+1\right)-2\\&=x-y-1\\&=d(x-y)\end{aligned}$$

for all $x,y\in\mathbb{Z}$, and so *d* is a (ℓ,r)-derivation of X. But

$$\begin{aligned}\left(1-d(0)\right)\wedge\left(d(1)-0\right)&=\left(1-(-1)\right)\wedge\left(0-0\right)=2\wedge 0=0-(0-2)\\&=2\neq 0=d(1)=d(1-0)\end{aligned}$$

and thus d is not a (r,ℓ)- derivation of X .

Definition 3.4. A self map d of a *B* – algebra X is said to be regular if $d(0)=0$.

Proposition 3.5. Let *d* be a (ℓ,r)- derivation of *B*-algebra X . Then

(i) $d(0)=d(x)*x$ for all $x\in X$,

(ii) *d* is 1-1.

(iii) If *d* is regular , then it is the identity map.

(iv) If there is an element $x\in X$ such that $d(x)=x$, then *d* is the identity map.

(v) If there is an element $x\in X$ such that $d(y)*x=0$ or $x*d(y)=0$ for all $y\in X$, then

$$d(y)=x \quad \text{for all } y\in X$$, i.e. *d* is constant.

Proof

(i) Let $x\in X$. Then $x*x=0$ and so

$$\begin{aligned}d(0)=d(x*x)&=\left(d(x)*x\right)\wedge\left(x*d(x)\right)=\left(x*d(x)\right)*\left[\left(x*d(x)\right)*\left(d(x)*x\right)\right]\\&=\left[\left(x*d(x)\right)*\left(0*\left(d(x)*x\right)\right)\right]*\left(x*d(x)\right) && \text{by}(2)\\&=\left[\left(x*d(x)\right)*\left(x*d(x)\right)\right]*\left(x*d(x)\right) && \text{by}(8)\\&=0*\left(x*d(x)\right) && \text{by}(5)\\&=d(x)*x && \text{by}(8)\end{aligned}$$

(ii) Let $x, y \in X$ such that $d(x) = d(y)$. Then by (i), we have

$$d(0) = d(x) * x \ .$$

Also, by (i)

$$d(0) = d(y) * y \,.$$

Thus,

$$d(x) * x = d(y) * y \,.$$

Therefore,

$$d(x) * x = d(x) * y \,.$$

Using Theorem 2.4 we have,

$$x = y \ .$$

That is d is 1-1.

(iii) Suppose that d is regular, and $x \in X$, thus

$d(0) = 0$

$0 = d(x) * x$ by(i)

$d(x) = x$ by(3)

for all $x \in X$. That is d is the identity map.

(iv) Suppose $d(x) = x$ for some $x \in X$. Then

$d(x) * x = 0$ by (3)

$d(0) = 0$ by (i)

Using (iii)we have that d is the identity map.

(v) Follows directly from (3).

Proposition 3.6. Let d be (r, ℓ)- derivation of B-algebra X. Then

(i) $d(0) = x * d(x)$ for all $x \in X$.
(ii) $d(x) = d(x) \wedge x$ for all $x \in X$.
(iii) d is 1-1.
(iv) If d is regular , then it is the identity map.
(v) If there is an element $x \in X$ such that $d(x) = x$, then d is the identity map.

(vi) If there is an element $x \in X$ such that $d(y)*x=0$ or $x*d(y)=0$ for all $y \in X$, then

$$d(y)=x \quad \text{for all } y \in X, \text{ i.e. } d \text{ is constant.}$$

Proof

(i) Let $x \in X$. Then $x*x=0$ and so

$$d(0)=d(x*x)=\big(x*d(x)\big)\wedge\big(d(x)*x\big)=\big(d(x)*x\big)*\Big[\big(d(x)*x\big)*\big(x*d(x)\big)\Big]$$

$$=\Big[\big(d(x)*x\big)*\big(0*\big(x*d(x)\big)\big)\Big]*\big(d(x)*x\big) \quad \text{by(2)}$$

$$=\Big[\big(d(x)*x\big)*\big(d(x)*x\big)\Big]*\big(d(x)*x\big) \quad \text{by (8)}$$

$$=0*\big(d(x)*x\big) \quad \text{by(5)}$$

$$=x*d(x) \quad \text{by(8)}$$

(ii) Let $x \in X$. Then $x*0=x$ and so

$$d(x)=d(x*0)=\big(x*d(0)\big)\wedge d(x)=d(x)*\Big[d(x)*\big(x*d(0)\big)\Big]$$

$$= d(x)*\Big[d(x)*\big(x*\big(x*d(x)\big)\big)\Big] \quad \text{by(i)}$$

that is,

$$d(x)*0= d(x)*\Big[d(x)*\big(x*\big(x*d(x)\big)\big)\Big].$$

From Theorem 2.4 we obtain

$$d(x)*\Big[x*\big(x*d(x)\big)\Big]=0,$$

and from (3) we have

$$d(x)=\Big[x*\big(x*d(x)\big)\Big]$$

$$=d(x)\wedge x.$$

(iii) Let $x, y \in X$ such that $d(x)=d(y)$,then by (i)

$$d(0)=x*d(x).$$

Also, by (i)

$$d(0)=y*d(y).$$

Thus,

$$x*d(x)=y*d(y),$$

therefore,

$$x * d(x) = y * d(x).$$

Using Theorem 2.4 we have,

$$x = y .$$

That is d is 1-1.

(iv) Let d be regular, and $x \in X$. Thus

$$d(0) = 0$$

$$0 = x * d(x) \quad \text{by(i)}$$

$$d(x) = x \quad \text{by(3)}$$

for all $x \in X$. That is d is the identity map.

(v) Suppose $d(x) = x$ for some $x \in X$. Then

$$x * d(x) = 0 \quad \text{by (3)}$$

$$d(0) = 0 \quad \text{by (i)}$$

Using (iv) we have that d is the identity map.

(vi) Follows directly from (3).

4. Derivations of 0-commutative *B*-algebra

In this section we investigate derivation of 0-commutative *B*-algebra.

Example 4.1. Let $X = \{0, 1, 2, 3\}$ be a 0-commutative *B*-algebra with Cayley table (Table 2) as follows :

Table 2.

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a map $d : X \to X$ by :

$$d(x) = \begin{cases} 2 & if \quad x = 0, \\ 3 & if \quad x = 1, \\ 0 & if \quad x = 2, \\ 1 & if \quad x = 3. \end{cases}$$

Then it is easily checked that d is a derivation of X .

Proposition 4.2 Let $(X,*,0)$ be a 0-commutative *B*-algebra, and d is a (ℓ,r)-derivation of X. Then:

(i) $d(x*y)=d(x)*y$.

(ii) $d(x)*d(y)=x*y$.

for all $x,y\in X$.

Proof

(i) Let $x,y\in X$, then

$$d(x*y)=(d(x)*y)\wedge(x*d(y))$$
$$=(x*d(y))*\left[(x*d(y))*(d(x)*y)\right]$$
$$=d(x)*y \qquad \text{by(14)}$$

(ii) Let $x,y\in X$, then from Proposition 3.5 (i) we have:

$$d(0)=d(x)*x.$$

Also,

$$d(0)=d(y)*y,$$

thus,

$$d(x)*x=d(y)*y,$$

that is,

$$(d(y)*y)*(d(x)*x)=0$$
$$(x*y)*(d(x)*d(y))=0 \qquad \text{by (13)}$$
$$d(x)*d(y)=x*y \qquad \text{by (3)}$$

Similarly , we can prove:

Proposition 4.3 Let $(X,*,0)$ be a 0-commutative *B*-algebra, and d is a (r,ℓ)- derivation of X. Then:

(i) $d(x*y)=x*d(y)$.

(ii) $d(x)*d(y)=x*y$.

Definition 4.4 Let X be a B-algebra and d_1, d_2 two self maps of X. We define $d_1 \circ d_2 : X \to X$ as :

$$d_1 \circ d_2(x) = d_1(d_2(x)) \text{ , for all } x \in X .$$

Proposition 4.5 Let X be a 0-commutative B-algebra and d_1, d_2 (ℓ, r)-derivations of X. Then $d_1 \circ d_2$ is also a (ℓ, r)-derivation of X.

Proof

Let X be a 0-commutative B-algebra and d_1, d_2 are (ℓ, r)- derivation of X, and let $x, y \in X$. Then:

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1\left[(d_2(x) * y) \wedge (x * d_2(y))\right] \\ &= d_1(d_2(x) * y) \qquad \text{by(14)}\end{aligned}$$

Using proposition 4.2(i), we get:

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1(d_2(x)) * y \\ &= (x * d_1(d_2(y))) * \left[(x * d_1(d_2(y))) * (d_1(d_2(x)) * y)\right] \qquad \text{by(14)} \\ &= ((d_1 \circ d_2)(x) * y) \wedge (x * (d_1 \circ d_2)(y)).\end{aligned}$$

Which implies that $d_1 \circ d_2$ is a (ℓ, r)- derivation of X.

Similarly, we can prove:

Proposition 4.6 Let X be a 0-commutative B-algebra and let d_1, d_2 be (r, ℓ) - derivations of X. Then $d_1 \circ d_2$ is also a (r, ℓ)-derivation of X.

Combining Propositions 4.5 and 4.6 , we get :

Theorem 4.7 Let X be a 0-commutative B-algebra and let d_1, d_2 be derivations of X. Then $d_1 \circ d_2$ is also a derivation of X.

Proposition 4.8 Let X be a 0-commutative B-algebra and d_1, d_2 be derivations of X. Then $d_1 \circ d_2 = d_2 \circ d_1$.

Proof

Let X be a 0-commutative *B*-algebra and d_1, d_2 are derivations of X. Since d_2 is a (ℓ,r)- derivation of X, then from proposition 4.2(i), we have:

$$(d_1 \circ d_2)(x*y) = d_1(d_2(x*y)) = d_1(d_2(x)*y)$$

for all $x, y \in X$.

But d_1 is (r,ℓ)- derivation of X, so from proposition 4.3(i) we obtain:

$$d_1(d_2(x)*y) = d_2(x)*d_1(y)$$

thus we have for all $x, y \in X$, we have

$$(d_1 \circ d_2)(x*y) = d_2(x)*d_1(y) \tag{15}$$

Also, since d_1 is (r,ℓ)- derivation of X, we have

$$(d_2 \circ d_1)(x*y) = d_2(x*d_1(y)), \quad \text{for all } x, y \in X.$$

But d_2 is (ℓ,r)- derivation of X, so

$$d_2(x*d_1(y)) = d_2(x)*d_1(y), \quad \text{for all } x, y \in X.$$

Thus, for all $x, y \in X$, we have

$$(d_2 \circ d_1)(x*y) = d_2(x)*d_1(y) \tag{16}$$

From (15) and (16) we get

$$(d_1 \circ d_2)(x*y) = (d_2 \circ d_1)(x*y), \quad \text{for all } x, y \in X.$$

Putting $y = 0$, we get

$$(d_1 \circ d_2)(x) = (d_2 \circ d_1)(x), \quad \text{for all } x, y \in X,$$

which implies that $d_1 \circ d_2 = d_2 \circ d_1$.

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الاشتقاقيات على جبور− B

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المستخلص. اتجهت الأبحاث المتخصصة في علم الجبر مؤخرًا إلى دراسة مفاهيم جبرية حديثة تمثلت في جبر− *BCK/BCI/BCH* وغير ذلك، كما يُعدُّ جبر−B أحد هذه الجبور والذي يعتبر محور الدراسة في هذا البحث. ومن أهم المواضيع التي تطرقت لها الأبحاث التي نشرت في دراسة هذه الجبور هي تطبيق راسم الاشتقاقية المعرف سابقاً على الحلقات على هذه الجبور حيث تم تعريف راسم الاشتقاقية من جبر−*BCI* إلى آخر ودراسة تأثير هذا الراسم على الجبر ومحاولة استخلاص النتائج المرتبطة به.

قدمنا في هذا البحث دراسة جديدة لراسم الاشتقاقية تتمثل في تعريف هذا الراسم على جبر-B الذي اعتمد على تعريف كلٌ من الاشتقاقية اليمنى− اليسرى والاشتقاقية اليسرى− اليمنى ودعم هذه التعاريف بالأمثلة، ومن ثم برهنة بعض الخصائص الجبرية لهاتين الاشتقاقيتين. بعد ذلك اتجهنا لدراسة تأثير هذا الراسم على نوع خاص من جبور−B وهو جبر−B الإبدالي−0 واستخلاص خواصه الجبرية عند تعريف كلٌ من الاشتقاقية اليمنى− اليسرى، والاشتقاقية اليسرى−اليمنى عليه. كما أثبتنا أن تحصيل اشتقاقيتين معرفتين على جبر−B الإبدالي−0 هو أيضًا اشتقاقية، وأن عملية تحصيل رواسم الاشتقاقية على جبر−B الإبدالي−0 هي إبدالية.