J.K.A.U.: Sci., vol. 1, pp. 221-227 (1409 A.H. / 1989 A.D.

A Note on the Axioms for a Length Function on a Group

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ABSTRACT. The author introduces the concept of a length function on a group satisfying only two axioms. Under such general length function, he investigates properties of the elements of the group. General results are obtained and a subgroup theorem is proved.

1. Introduction

The ideas of Nielsen[1], and in particular the idea of calculating the extent of cancellation in a product of reduced words in a free group, had motivated Lyndon[2] to develop a set of axioms for a length function from a group to the set non-negative integers. He postulated six axioms for integer – valued length functions on free groups and free products.

In this paper we define a general length function; a real-valued length function satisfying only two of Lyndon's axioms, and then obtain some consequences and results under such a length function.

2. Length Functions

We give the following definition:

2.1 Definition

Let G be a group. A length function $|\ |: G \to R$ is a function assigning to each element x in G a real number $|x|$ such that for all x, y, z in G the following axioms are satisfied:

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A_2. $|x^{-1}| = |x|$

A_4. $d(x,y) < d(x,z)$ implies $d(x,y) = d(y,z)$

where $d(x,y) = \frac{1}{2}[|x| + |y| - |xy^{-1}|]$.

That is, the two smallest numbers in the triple of real numbers $d(x,y)$, $d(x,z)$, $d(y,z)$ are equal. The numbering of the axioms is due to Lyndon[2].

An example of a length function is the usual modulus function on R, the additive group of the real numbers.

2.2 Proposition

For $x,y \in G$, we have

(i) $d(x,y) \geq \frac{1}{2}|1|$, 1 is the identity element of G

(ii) $|x| \geq |1|$

(iii) $d(x,y) \leq |x| - \frac{1}{2}|1|$.

Proof

(i) $2d(x,1) = |x| + |1| - |x| = |1| = 2d(1,y)$; hence by A_4 , $2d(x,y) \geq |1|$.

(ii) $2d(x,x) = |x| + |x| - |1| \geq |1|$, by (i) .

(iii)
$$\begin{aligned} 2d(x,y) &= |x| + |y| - |xy^{-1}| \\ &= 2|x| - (|x| + |xy^{-1}| - |y|) \\ &= 2|x| - 2d(yx^{-1}, x^{-1}) \\ &\leq 2|x| - |1| \end{aligned}$$

That is, $d(x,y) \leq |x| - \frac{1}{2}|1|$.

As another example, let any element $x = x_1 x_2 \ldots x_n$ as a reduced word in a free group F with basis X. Then $|x| = n$ defines a length function on F satisfying the above axioms.

If the length function is normalized by subtrating $|1|$, then it still satisfies A_2 and A_4; so without loss of generality, we may assume that the length function on G satisfies axiom:

$$A_1'. \ |1| = 0$$

This is a weakened form of Lyndon's axiom:

$$A_1 . \ |x| = o \text{ if and only } x = 1 \text{ in } G ;$$

but it may be noted here that A_1' is added more for convenience than necessity.

Using Proposition 2.2 and A_1', A_2, A_4 the consequences in the following Proposition are immediately obtained.

2.3 Proposition

(i) $d(x,y) \geq 0$; this is Lyndon's axiom A_3

(ii) $d(x,x) = |x| \geq 0$

(iii) $d(x,y) \leq |x|$

(iv) $d(x,y) = d(y,x)$.

3. Non-Archimedean Elements

3.1 Definition

In a group G with a length function an element x is archimedean if $|x^2| > |x|$ and non-archimedean if $|x^2| \leq |x|$.

Let $N = \{ x \in G \ : \ |x^2| \leq |x| \}$.

3.1 Proposition

Let x be an element of G. Then

(i) $x \in N$ implies $|x^n| \leq |x|$ for all integers $n \geq 0$,

(ii) $x \notin N$ implies $|x^n| = |x| + (n-1)\,t$ for any positive integer n and where $t = |x^2| - |x|$.

Proof

(i) Let $x \in N$. We use induction on n. Result holds trivially for $n = 0, 1$ and by definition for $n = 2$.

Suppose result is true for all non-negative integers $\leq n$.

That is, $|x^{n-1}|, |x^n| \leq |x|$.

Now $2d(x^n,x) = |x^n| + |x| - |x^{n-1}| \geq |x^n|$,

$2d(x,x^{-1}) = |x| + |x^{-1}| - |x^2| \geq |x| \geq |x^n|$;

hence by A_4, $2d\,(x^n, x^{-1}) \geq |x^n|$.

That is, $|x^n| + |x| - |x^{n+1}| \geq |x^n|$, giving $|x^{n+1}| \leq |x|$.

Hence result holds for all integers $n \geq 0$.

(ii) For $x \notin N$, the formula $|x^n| = |x| + (n-1)\,t$ holds for $n = 1,2$. Assume result for all positive integers $\leq n$.

That is, $|x^{n-1}| = |x| + (n-2)t$, $|x^n| = |x| + (n-1)\,t$.

We have, $2d(x^n,x) = |x^n| + |x| - |x^{n-1}| = |x| + t$,

$2d(x,x^{-1}) = |x| + |x| - |x^2| = |x| - t$.

For $x \notin N$, $t > 0$ and therefore $|x| - t < |x| + t$.

Thus $2d(x,x^{-1}) < 2d(x^n,x)$, and hence by A_4,

We have $2d(x^n,x^{-1}) = 2d(x,x^{-1})$.

That is, $|x^n| + |x| - |x^{n+1}| = |x| - t$,

hence $|x^{n+1}| = |x^n| + t = |x| + nt$.

Hence results holds for all positive integers n.

The Proposition implies that an element x of finite order G is non-archimedean, and if x is archimedean then the lengths $|x^n|$ are unbounded, a result obtained in[2].

3.1.1 Corollary

$x \in N$ implies $yxy^{-1} \in N$ for all $y \in G$.

Proof

$$2d(yx^n,y) = |yx^n| + |y| - |yx^ny^{-1}| \quad (1)$$

$$2d(y,x^{-n}) = |y| + |x^n| + |yx^n| \quad (2)$$

From (1) and (2) respectively, we have

$$|yx^ny^{-1}| \leq |yx^n| + |y| \text{ and } |yx^n| \leq |y| + |x^n|.$$

By part (i) of the Proposition, $|x^n| \leq |x|$ for all $n \geq 0$.

Therefore, $|yx^ny^{-1}| \leq |x| + 2|y|$.

That is, the lengths $|(yxy^{-1})^n|$ are bounded and hence $y x y^{-1} \in N$.

3.2 Proposition

$K = \{ x \in G : |x| = 0 \}$ is a subgroup contained in N.

Proof

Clearly K is contained in N. K is a subgroup since if $|x| = |y| = 0$ then $-\frac{1}{2}|xy^{-1}| = d(x,y) \geq 0$;

therefore, $|xy^{-1}| = 0$.

3.2 Definition

A relation $\sim$ on G is defined by $x \sim y$ if and only if $|xy^{-1}| \leq |x| = |y|$ or equivalently by $x \sim y$ if and only if $2d(x,y) \geq |x| = |y|$.

This is an equivalence relation on G; it is clearly reflexive and symmetric and transitivity follows from A_4.

Let C(x) denote the equivalence class of $x \in N$, and $M(x) = \{ yz : y,z \in C(x)\}$.

3.3 Proposition

$$x \sim x^{-1} \sim y \sim y^{-1} \text{ implies that } xy \in N$$

Proof

For any element x of G, $x \sim x^{-1}$ if and only if $x \in N$.

As the elements x, x^{-1}, y, y^{-1} are all equivalent, we have

$$|xy^{-1}|, |xy|, |yx| \leq |x| = |x^{-1}| = |y| = |y^{-1}| .$$

$$2d(xy,y) = |xy| + |y| - |x| = |xy| ,$$

$$2d(y,x^{-1}) = |y| + |x| - |yx| = 2|x| - |yx| ,$$

$$2d(xy,x^{-1}) = |xy| + |x| - |xyx| .$$

Since $|xy|, |yx| \leq |x|$ it follows that $d(xy,y) \leq d(y,x^{-1})$, and hence by A_4, $d(xy,y) \leq d(xy,x^{-1})$, giving $|xyz| \leq |x|$.

Now $2d(xy,y^{-1}x^{-1}) = |xy| + |xy| - |(xy)^2|$,

$$2d(y^{-1}x^{-1},x^{-1}) = |xy| + |x| - |y| = |xy|,$$

$$2nd(xy,x^{-1}) = |xy| + |x| - |xyx|.$$

As $|xyx| \leq |x|$ it follows that $d(y^{-1}x^{-1},x^{-1}) \leq d(xy,x^{-1})$ and hence by A_4, $d(y^{-1}x^{-1},x^{-1}) \leq d(xy,y^{-1}x^{-1})$ giving $|(xy)^2| \leq |xy|$. Thus $xy \in N$.

3.1 Theorem

$H = C(x) \cup M(x)$ is a subgroup of G.

Proof

By definition C(x) is a subset of N and, by Proposition 3.3 M(x) is also contained in N; therefore H is contained in N. Since C(x) contains the inverses of its elements, the same is true for M(x), which contains the identity element 1. Thus it remains to show that H is closed under multiplication. For this purpose we consider products of elements in C(x). Let $x,y,z \in C(x)$, then if $|xy| = |x| = |y|$ it follows from definition 3.2 that $xy \sim y$ and hence $xy \sim C(x)$, and so $xyz \in M(x)$. If $|xy| < |x| = |y| = |z|$ then $2d(xy,y) < 2d(y,x^{-1})$, and A_4 implies $d(xy,y) = d(xy,x^{-1})$, giving $|xyx| = |x|$; and we consider

$$2d(xyx,x) = |xyx| + |x| - |xy| = 2|x| - |xy|,$$

$$2d(z^{-1}x,x) = |z^{-1}x| + |x| - |z^{-1}| = |z^{-1}x|.$$

Since $x \sim x^{-1} \sim z \sim z^{-1}$, we have $|z^{-1}x| \leq |z^{-1}| = |x^{-1}| = |x|$.

Hence $d(z^{-1}x,x) < d(xyx,x)$ and A_4 implies that $d(z^{-1}x,x) = d(xyx,z^{-1}x)$, giving $|xyz| = |x|$.

Thus $|xy| = |xyzz^{-1}| < |xyz| = |x| = |z|$. By definition 3.2, this shows that $xyz \sim z \sim$

x and xyz $\in$ C(x). Also, if x,y,z,w $\in$ C(x) then, if $|xy| = |x| = |y|$, we have shown above that xy $\in$ C(x), and so xyzw is the product of three elements from C(x) which, as we have shown is an element of C(x) $\cup$ M(x). If $|xy| < |x| = |y|$, then we have shown that xyz $\in$ C(x), and so xyzw $\in$ M(x). These cases cover the product of two elements of H = C(x) $\cup$ M(x), proving that H is a group.

Acknowledgement

The author is grateful to the referees for their comments and suggestions.

References

[1] **Nielsen, J.**, A basis for subgroups of free groups, *Math. Scand.*, **3:** 31-43 (1955).
[2] **Lyndon, R.C.**, Length functions in groups, *Math. Scand.*, **12:** 209-234 (1963).

دراســــة في مسلمات دالة الطــول على الزمـــر

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يدرس هذا البحث مفهوم دالة الطول على الزمر حيث يقدم دالة طول عامة تحقق مسلمتين فقط ومعرفة على أية زمرة .

ثم يدرس البحث خواص عناصر الزمرة تحت تأثير دالة الطول هذه ، حيث يصل إلى نتائج عامة تصنف عناصر الزمرة ، ومن ثم يبرهن نظريةً لزمرة جزئية في الزمرة الأصلية .

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