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On the Strong Limit of Well - Bounded Operators

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In this paper we prove that the strong limit of a net (S_α) of well-bounded operators is a well-bounded operator S. Also we prove that the strong limit of $f(S_\alpha)$ is $f(S)$ whenever $f \in AC(J)$.

1. Notations

Throughout this paper, X is a complex Banach space, and L(X) is the algebra of all bounded linear operators on X. If J=[a,b] is a compact interval of the real line R, then we use BV(J) to denote the Banach algebra of all complex-valued functions of bounded variation on J with norm $|||.|||$ defined by

$$|||f||| = |f(b)| + \text{var}\,(f,J) \qquad (f \in BV(J))$$

where var (f,J) is the total variation of f over J. The Banach subalgebra of BV(J) consisting of all absolutely continuous functions on J will be denoted by AC(J). We use $\underset{\sim}{P}(J)$ to denote the subalgebra of AC(J) consisting of all polynomials on J.

2. Well-bounded operators

2.1. Definition

Let $T \in L(X)$. We say that T is well-bounded if there is a compact interval J and a real constant K such that

$$\| P(T) \| \leq K\,|||P||| \qquad (P \in \underset{\sim}{P}(J)) \qquad (2.1.1).$$

For a full discussion of well-bounded operators, the reader is advised to see (Dowson, 1978).

2.2. Theorem

Let $\{T_\alpha\}$ be a bounded net of bounded linear operators on an arbitrary Banach space X such that $\{T_\alpha\}$ converges to T in the strong operator topology ($T_\alpha \xrightarrow{S} T$). Then $T_\alpha^n \xrightarrow{S} T^n$ for any positive interger n.

Proof.

Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two bounded nets of bounded linear operators on X such that:

$$A_\alpha \xrightarrow{S} A,\ B_\alpha \xrightarrow{S} B \text{ and } \| A_\alpha \| \leq K$$

Then for $x=x_1, \ldots, x_n \in X$, we have:

$$\| (A_\alpha - A)Bx \| \leq \frac{\varepsilon_1}{2} \text{ and } | (B_\alpha - B)x | \leq \varepsilon_2$$

for a sufficiently large α and for an arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Now

$$\| (A_\alpha B_\alpha - AB) \| = \| (A_\alpha B_\alpha - A_\alpha B + A_\alpha B - AB)\, x \|$$

Using the triangle inequality we get

$$\| (A_\alpha B_\alpha - AB)\, x \| \leq \| A_\alpha (B_\alpha - B)\, x \| + \| (A_\alpha - A)\, Bx \|$$

$$\leq K \| (B_\alpha - B)\, x \| + \frac{\varepsilon_1}{2}$$

Taking $\varepsilon_2 = \dfrac{\varepsilon_1}{2K}$ we get

$$\| (A_\alpha B_\alpha - AB)\, x \| \leq K. \frac{\varepsilon_1}{2K} + \frac{\varepsilon_1}{2} = \varepsilon_1$$

Hence $A_\alpha B_\alpha \xrightarrow{S} AB$ which means that $T_\alpha^2 \xrightarrow{S} T^2$ whenever $T_\alpha \xrightarrow{S} T$ and by induction the proof is complete.

3. Theorem

Let X be a Banach space and let $\{S_\alpha : \alpha \in A\}$ be a net of well-bounded operators on X converging to S in the strong operators topology such that $\| S_\alpha \| < M < \infty$.

Then

(1) S is a well-bounded operator.

(2) if $J = [a,b]$ is a compact interval satisfying the conditions in Definition 2.1 and if $f \in AC(J)$, then $f(S_\alpha) \xrightarrow{S} f(S)$.

Proof

(1) Let P be any complex polynomial. Since S_α, $(\alpha \in A)$, is well bounded then,

$$\| P(S_\alpha) \| \leq K \, ||| P |||$$

Since $S_\alpha \xrightarrow{S} S$, then, by Theorem 2.2., $S_\alpha^n \xrightarrow{S} S^n$. Hence $P(S_\alpha) \xrightarrow{S} P(S)$. This implies that

$$\| P(S) \| \leq K \, ||| P |||,$$

Which means that S is well-bounded.

(2) Let $f \in AC(J)$. Since polynomials are norm dense in AC(J), there exists a sequence $\{P_n\}$ of polynomials such that $P_n \to f$ in the norm of AC(J). Since S_α is well-bounded for all α (in the index set A), we have:

$$\| P_n(S_\alpha) \| \leq K \, ||| P_n ||| = K(|P_n(b)| + \text{var}\,(P_n, J)).$$

Since $P_n \to f$, we have

$$\| f(S_\alpha) \| \leq K \, ||| f |||$$

Hence,

$$\| P_n(S_\alpha) - f(S_\alpha) \| \leq K \, ||| P_n - f |||$$
$$\leq K \left\{ |P_n(b) - f(b)| + \text{var}\,(P_n - f) \right\}$$

Hence

$$P_n(S_\alpha)\, x \to f(S_\alpha)\, x \text{ uniformly for every } \| x \| \leq 1.$$

Thus, by the Moore-smith convergence theorem (Dunford and Schwartz, 1958).

$$f(S)x = \lim_n \lim_\alpha P_n(S_\alpha)x = \lim_\alpha \lim_n P_n(S_\alpha)x = \lim_\alpha f(S_\alpha)x.$$

Hence $$f(S_\alpha) \xrightarrow{S} f(S) \qquad (f \in AC(J)).$$

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النهايات القوية للمؤثرات المحدودة جيداً

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في هذا البحث نثبت أن النهاية القوية لشبكة $س_{ن}$ من المؤثرات المحدودة جيداً هي مؤثر محدود جيداً س. كذلك نثبت أن النهاية القوية لـ ق($س_{ن}$) هي ق (س) حيث ق تنتمي لمجموعة الدوال مطلقة الاتصال على فترة مغلقة ل.