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Some Limiting Results

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This paper considers $\lim_{n\to\infty} n^{-a} \sum_{J=0}^{n} \sum_{x} A_{n-j}(x)\, B_j(x)$ and $\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \int A_{n-j}(x)\, B_j(x)\, dx$ and their application in the Central Limit theorem. To arrive at these results some ratio limit theorems for numbers are generalized for functions. It has been proved that under certain conditions these limits are equal to the limits $\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{x} A_j(x)\, B(x)$ and $\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \int A_j(x)\, B(x)\, dx$ respectively where $B(x) = \lim_{n\to\infty} B_n(x)$.

Central limit theorem plays a great role in the theory of statistics. Chebyshev and Markov proved that if

$$\lim_{n\to\infty} E(Y_n) = \begin{cases} 1.\ 3.\ 5.\ \ldots .(k-1), & \text{when } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where $Y_n = \sum_{i=1}^{n} (X_i - \mu_i)/\sqrt{n}$ with $\mu_j = E(X_j)$ and $X_1, X_2, \ldots$ is a sequence of random variables, then in the limit Y_n has normal distribution with mean zero and variance $\delta^2 = \lim_{n\to\infty} \mathrm{Var}(Y_n)$. Thus to prove the central limit theorem under certain given conditions one has to prove a set of limiting results which can be used in proving the central limit theorem. To start with we first state two lemmas the proof of which can be found in most text books. We are quoting the first one from Karlin (1966, p127) and the second from Chung (1967, p66).

Lemma 1. If $0 \leq a_n \leq K$ and $\sum_{n=0}^{\infty} a_n = \infty$, then the relation $\lim_{n\to\infty} b_n = b$ with b finite implies

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} a_{n-j} b_j}{\sum_{j=0}^{n} a_j} = b.$$

Lemma 2. If $0 \leq a_n \leq K$, a sufficient condition that

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} a_{n-j} b_j}{\sum_{j=0}^{n} a_j} = \lim_{n\to\infty} b_n$$

whenever $\{b_n\}$ has a limit is that $\lim_{n\to\infty} \dfrac{a_n}{\sum_{j=0}^{n} a_j} = 0.$

It is obvious that condition $\lim_{n\to\infty} \dfrac{a_n}{\sum_{j=0}^{n} a_j}$ is linient as compared to the condition $\lim_{n\to\infty} \sum_{j=0}^{n} a_j = \infty$ and that the latter implies the former but not conversely.

In this paper our aim is to generalize these lemmas when we have functions $A_j(x)$ and $B_j(x)$ instead of a_j and b_j. Section 2 is devoted to the situation when A(x) and B(x) are functions of the discrete variable x. In this case we will generalize lemma 1. The continuous case will be considered in section 3 and there we will generalize lemma 2.

2. *Discrete case*

Let $A_j(x)$ and $B_j(x)$ be defined on I, the set of natural numbers, for all j. Suppose further that the sum $\sum_X A_j(x)$ exists which we may denote by a_j. We are going to generalize lemma 1 in the following form.

Theorem 1. Let $\{A_n(x)\}$ be a uniformly bounded positive sequence such that $\lim_{n\to\infty} \sum_{j=0}^{n} \sum_X A_j(x) = \infty$. Let the sequence $\{B_n(x)\}$ converge uniformly to the bounded function B(x) such that $\min_X \{|B(x)|\} = b > 0$. Then

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)} = 1$$

provided B (x) does not change sign in the domain of definition.

Proof. Since $B_n(x)$ converges to B (x) uniformly in x, there exists an N (depending on a given $\varepsilon > 0$) such that

$$| B_n(x) - B(x) | < b\,\varepsilon \qquad \text{for all } n > N.$$

Moreover $B_n(x)$ converges to the bounded function B (x), there exists an M such that $|B_n(x)| \leq M$ for all n and $|B(x)| \leq M$. Further B(x) does not change sign, we have

$$\left| \sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x) \right| \geq b \sum_{j=0}^{n} \sum_{X} A_j(x)$$

Now

$$\left| \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)} - 1 \right| = \left| \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x) - B(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)} \right|$$

$$\leq \frac{\sum_{j=0}^{N} \sum_{X} A_{n-j}(x) \left| B_j(x) - B(x) \right|}{\left| \sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x) \right|} + \frac{\sum_{j=N+1}^{n} \sum_{X} A_{n-j}(x) \left| B_j(x) - B(x) \right|}{\left| \sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x) \right|}$$

X,

$$< \frac{2M}{b} \frac{\sum_{j=0}^{N} a_{n-j}}{\sum_{j=0}^{n} a_j} + \frac{b\varepsilon}{b} \frac{\sum_{j=N+1}^{n} a_{n-j}}{\sum_{j=0}^{n} a_j}$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$\lim_{n \to \infty} \left| \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)} - 1 \right| < \varepsilon$$

But ϵ is arbitrary which shows that the limit is zero and this proves the theorem.

We note that if $\lim_{n\to\infty} B_n(x) = B$, a constant, then

$$\lim_{n \to \infty} \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)} = B$$

In this case B need not be other than zero.

We are now going to prove another theorem which is based on theorem 1 and is most helpful in proving the central limit theorem.

Theorem 2. Let $\{ A_n(x) \}$ be a uniformly bounded positive sequence such that $\lim_{n\to\infty} \sum_{j=0}^{n} \sum_{X} A_j(x) = \infty$. Let the sequence $\{ B_n(x) \}$ converges uniformly to the bounded function $B(x)$ such that $\min \{ |B(x)| \} = b > 0$. Then

$$\lim_{n\to\infty} \sum_{j=0}^{n} n^{-a} \sum_{X} A_{n-j}(x)\, B_j(x) = \lim_{n\to\infty} \sum_{j=0}^{n} n^{-a} \sum_{X} A_j(x)\, B(x) \qquad (1)$$

provided the limit on the right hand side exists for some $a > 0$ and that $B(x)$ does not change sign in the domain of definition.

Proof. We have

$$\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x) = \lim_{n\to\infty} \left\{ \frac{\sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x)}{\sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)} \right\} . n^{-a} \sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)$$

Using theorem 1 and the existence of $\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{X} A_j(x)\, B(x)$, we obtain the required result.

If B(x) is independent of x, then with B(x) – B, say, we have

$$\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\, B_j(x) = B \lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{X} A_j(x).$$

If $\lim_{n\to\infty} a_n = A$, say, then obviously

$$\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \sum_{X} A_{n-j}(x)\; B_j(x) = \begin{cases} AB & \text{if } a = 1 \\ 0 & \text{if } a > 1. \end{cases}$$

To show an application of the above theorem we consider a Markov chain with finite state space and transition probability matrix $P = \{p_{ij}\}$ where $p_{ij} = \Pr(X_{m+1} = j \mid X_m = i)$ for all m and $\{\pi_x\}$ as its stationary distribution with μ as the mean and prove the

Theorem 3, If $X_1, X_2, \ldots, X_n, \ldots$ form a homogeneous Markov chain which is recurrent, irreducible and aperiodic, then

$$\lim_{n\to\infty} \operatorname{Var}\left(\sum_{j=1}^{n} X_j / \sqrt{n}\right) = \delta^2 - 2 \sum_{x} \sum_{y} (x-\mu)(y-\mu)\, \pi x\, \pi y\, \mu\, xy$$

Where μ_{xy} is the mean first entrance time from E to y and σ^2 is the variance of the stationary distribution.

Proof. Let $\Pr(X_0 = a) = 1$ and denote $E(X_n | X_0 = a)$ by $\mu_a(n)$ which we may write as μ_n if there is no fear of confusion. Let X be the random variable having the stationary distribution $\{\pi_i\}$ of the given Markov chain which is guaranteed by the given conditions. We have

$$\operatorname{Var}\left(\sum_{j=1}^{n} X_j / \sqrt{n}\right) = E\left[\sum_{j=1}^{n} (X_j - \mu_j) \Big/ \sqrt{n}\right]^{2}$$

$$= \sum_{j=1}^{n} n^{-1} E(X_j - \mu_j)^2 + 2n^{-1} \sum_{j=1}^{n} \sum_{i=1}^{j-1} E[(X_j - \mu_j)(X_i - \mu_i)]. \quad (2)$$

But

$$\sum_{j=1}^{n} \sum_{i=1}^{j-1} E[(X_j - \mu_j)(X_i - \mu_i)] = = \sum_{i=1}^{n} \sum_{j=1}^{n-i} E[(X_{j+i} - \mu_{j+i})(X_i - \mu_i)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n-i} E[(X_{j+i} - \mu_j^{*})(X_i - \mu_i)]$$

$$= \sum_{i=1}^{n} \sum_{x} (x - \mu_i)\, p_{ax}(i) \sum_{j=1}^{n-i} \sum_{y} (y - \mu_j^{*})\, P_{xy}(j)$$

$$= \sum_{i=1}^{n} \sum_{x} B_i(x)\, A_{n-i}(x) - \sum_{i=1}^{n} \sum_{x} C_i(x)\, A_{n-i}(x)$$

where $\mu_j^\bullet = \min_x E(X_j \mid X_o = x) = \mu_b(j)$, say, $B_m(x) = x\, p_{ax}(m)$,

$$C_m(x) = \mu_m\, p_{ax}(m) \text{ and } A_m(x) = \sum_{j=1}^{m} \sum_y (y - \mu_m^\bullet)\, p_{xy}(m).$$

With $E(X_j - \mu_j)^2 = \sigma_j^2$, we get from (2)

$$\lim_{n\to\infty} \mathrm{Var}\left(\sum_{j=1}^{n} X_j / \sqrt{n}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} n^{-1} \sigma_j^2 + 2 \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sum_x B_i(x)\, A_{n-i}(x)$$

$$- 2 \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sum_x C_i(x)\, A_{n-i}(x) \quad (3)$$

Since $\lim_{n\to\infty} B_n(x) = x\, \pi_x$ and $\lim_{n\to\infty} C_n(x) = \mu\, \pi_x$ and are both positive, therefore theorem 2 applies in (3) if we show that $\{A_n(x)\}$ is bounded and that $\sum_{n=1}^{\infty} \sum_x A_n(x) = \infty$. To do this it is enough to prove that $\lim_{n\to\infty} A_n(x)$ exists and is not zero. To this end we have, from Chung (1967, p66),

$$\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_y \{p_{xy}(j)\, p_{by}(j)\} . (y-\mu)$$

$$= \sum_y \left\{ \lim_{n\to\infty} \sum_{j=1}^{n} \left[p_{xy}(j) - p_{by}(j) \right] \right\} (y-\mu)$$

$$= \sum_y (\mu_{by} - \mu_{xy})\, (y-\mu)_{\pi y}$$

which is bounded and non-zero for positive recurrent chain. Therefore from (3) we obtain, using the validity of the interchange of two summations and of limit and summation,

$$\lim_{n\to\infty} \mathrm{Var}\left(\sum_{j=1}^{n} X_j / \sqrt{n}\right) = \sigma^2 + 2 \sum_x x\, \pi_x \left\{ \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} A_i(x) \right\}$$

$$- 2 \sum_X \mu\, \pi_X \left\{ \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} A_i(x) \right\}$$

$$= \sigma^2 - 2 \sum_x \sum_y (x - \mu)\, (y - \mu)_{\pi_x \pi_y \mu_{xy}}$$

and the proof is complete.

3. Continuous case

Let $\{A_n(x)\}$ and $\{B_n(x)\}$ be function sequences defined on a common domain D of the real line. Suppose that $\{A_n(x)\}$ and the double sequence $\{A_n(x) B_m(x)\}$ are Riemann integrable with respect to x. We are going to generalize lemma 2 in the following.

Theorem 4. Let $\{A_n(x)$ be a uniformly bounded non-negative sequence and suppose $\{B_n(x)\}$ converges uniformly to a bounded function B(x) which does not change sign in D and $\inf_{x \in D} \{ |B(x)| \} = b > 0$.

A sufficient condition that

$$\lim_{n \to \infty} \frac{\sum_{j=0}^{n} \int_D B_j(x) A_{n-j}(x)\,dx}{\sum_{j=0}^{n} \int_D B(x) A_j(x)\,dx} = 1$$

is that

$$\lim_{n \to \infty} \frac{\int_D A_n(x)\,dx}{\sum_{j=0}^{n} \int_D A_j(x)\,dx} = 0 \tag{4}$$

Proof. Since $B_n(x)$ converges uniformly to B(x), given any $\varepsilon > 0$ there exists an N such that $|B_n(x) - B(x)| < b\,\varepsilon$ for $n > N$ and all x. Moreover B(x) is bounded, there exists an M such that $|B_n(x) - B(x)| \leq M$ for all n and x. We observe that (4) also implies

$$\lim_{n \to \infty} \frac{\sum_{j=n-N}^{n} \int_D A_n(x)\,dx}{\sum_{j=0}^{n} \int_D A_j(x)\,dx} = 0. \tag{5}$$

Hence

$$\left| \frac{\sum_{j=0}^{n} \int_D B_j(x) A_{n-j}(x)\,dx}{\sum_{j=0}^{n} \int_D B(x) A_j(x)\,dx} - 1 \right| \leq \frac{\sum_{j=0}^{n} \int_D A_j(x) |B_{n-j}(x) - B(x)|\,dx}{\sum_{j=0}^{n} \int_D |B(x)|\, A_j(x)\,dx}$$

$$\leq \frac{1}{b \sum_{j=0}^{n} \int_D A_j(x)\,dx} \left\{ \sum_{j=0}^{n=n-1} \int_D A_j(x) \left|B_{n-j}(x) - B(x)\right| dx \right.$$

$$+ \sum_{j=n-N}^{n} \int_D A_j(x) \left| B_{n-j}(x) - B(x) \right| dx \Bigg\}$$

$$< \frac{b\varepsilon}{b} \frac{\sum_{j=0}^{n-N-1} \int_D A_j(x)\, dx}{\sum_{j=0}^{n} \int_D A_j(x)\, dx} + \frac{M}{b} \frac{\sum_{j=n-N}^{n} \int_D A_j(x)\, dx}{\sum_{j=0}^{n} \int_D A_j(x)\, dx}$$

Taking the limit on both the sides and using (5) we obtain

$$\lim_{n\to\infty} \left| \frac{\sum_{j=0}^{n} \int_D B_j(x) A_{n-j}(x)\, dx}{\sum_{j=0}^{n} \int_D B(x) A_j(x)\, dx} - 1 \right| < \varepsilon$$

Since ϵ is arbitrary, we conclude that the limit is zero which implies the theorem. It is to be noted that condition is satisfied if

a) $\lim_{n\to\infty} \sum_{j=0}^{n} \int_D A_j(x)\, dx < \infty$

or

b) $\lim_{n\to\infty} \sum_{j=0}^{n} \int_D A_j(x)\, dx = \infty$ and $\int_D A_n(x)\, dx$ is bounded.

We now state a theorem similar to theorem 2 for the continuous case and omit the proof. The proof goes on the same lines as that of theorem 2 with the exception that summation will be replaced by integration.

Theorem 5. If the conditions of theorem 4 are satisfied, then

$$\lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \int_D A_{n-j}(x)\, B_j(x)\, dx = \lim_{n\to\infty} n^{-a} \sum_{j=0}^{n} \int_D A_j(x)\, B(x)\, dx$$

Provided the limit on the right hand side exists for some $a > 0$.

It is obvious from this theorem that if $\lim_{n\to\infty} \int_D A_n(x)\, B(x)\, dx$ exists, then Cauchy's first theorem applies and for $a=1$ we have

$$\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n} \int_D A_{n-j}(x)\, B_j(x)\, dx = \lim_{n\to\infty} \int_D A_n(x)\, B(x)\, dx$$

Moreover if the sequence $\{A_n(x)\}$ is uniformly convergent to $A(x)$, then we have

$$\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n} \int_D A_{n-j}(x)\, B_j(x)\, dx = \int_D A(x)\, B(x)\, dx. \tag{6}$$

Finally we mention that a result similar to theorem 3 can be proved in the continuous case also. But it involves the concept of continuous state space Markov processes and therefore will be taken somewhere else.

Remark. It is clear that all our results in this paper are either the generalization of lemma 1 and lemma 2 or depend on these generalizations. However it will be worth while to mention that one can obtain results (1) and (6) by generalizing Cesaro's theorem whose special case is Cauchy's first theorem.

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بعض النتائج المحددة

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كلية العلوم بجامعة الملك سعود

تدرس في هذا البحث نهاية المقدارين $n^{-a} \sum_{j=0}^{n} \sum_{x} A_{n-j}(x) B_1(x)$ ، $n^{-a} \sum_{j=0}^{n} \int A_{n-j}(x) B_j(x)\, dx$

عندما تؤول قيمة n إلى ما لا نهاية، وتطبيقاتها في نظرية النهاية المركزية للوصول إلى هذه النتائج قمنا بتصميم بعض نظريات النهاية لنسب أعداد إلى نسب دوال. وأثبتنا تحت بعض الشروط أن هذه النهايات تساوي نهاية المقدارين $n^{-a} \sum_{j=0}^{n} \sum_{x} A_j(x) B(x)$ ، $n^{-a} \sum_{j=0}^{n} \int A_j(x) B(x)\, dx$ عندما تؤول قيمة n إلى ما لا نهاية، وعلى الترتيب، حيث أن $B(x)$ هي نهاية $B_n(x)$ عندما تؤول قيمة n إلى ما لا نهاية.