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Solving the One-Dimensional Neutron Transport Equation Using Chebyshev Polynomials and the Sumudu Transform - Part I

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Abstract : In this paper the Chebyshev polynomials and the Sumudu transform are combined to solve analytically the neutron transport equation in one-dimensional case. The procedure is based on the expansion of the angular flux in terms of the Chebyshev polynomials. The resulting system of linear differential equation is solved analytically using the Sumudu Transform technique.

1- Introduction:

The neutron transport equation is a linear case of the Boltzmann equation with wide applications in physics and engineering.

As is well known, the study of a given transport equation is a quite important and interesting in transport theory. Various methods have been developed to investigate, and special attention has been given to the task of searching methods that generate accurate results to transport problems in the context of deterministic methods based on analytical procedures, for the multidimensional transport problems. One of the effective methods to treat linear transport equation is the spectral method (Kim, Arnold. D. and Ishimaru, Akira, 1999; Kadem, A, 2006; W. Greenberg, C. Van der Mee and V. Protopopescu, 1987) whose basic goals is to find exact solution for approximations of the transport equation, several approaches have been suggested. Among them, the method proposed by (S. Chandrasekhar, 1960) solves analytically the discrete equations, *SN* equations, the spherical harmonics method (Duderstadt, J. J.; Martin, W. R, 1975) expands the angular flux in Legendre polynomials, the *FN* method (Garcia, R. D. M, 1985) transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain (Ganapol, B. D, et al,1994; Ganapol, B. D, 1992) the *SGF* method (Barros, R.C. and Larsen, E.W, 1991; Barros, R.C. and (Barichello, L .B.; Vilhena M.T., 1993) Larsen, E.W, 1990) is a numerical nodal method that generates numerical solution for the *SN* equations in slab geometry that is completely free of spatial truncation error. The *LTSN* method (M.T.Vilhena et al, 1991) solve analytically the *SN* equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the *LTSN* method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as LTSN (Barichello, L .B.; Vilhena M.T, 1993), *LTPN* (M.T.Vilhena., Streck, E. E, 1993), *LTChN* (Cardona , A. V.; M.T.Vilhena, 1994), *LTAN* (Cardona , A. V.; M.T.Vilhena,1997), *LTDN* (Barros, R.C. Cardona , A. V.; M.T.Vilhena, 1996).

The analytical character of this solution, in the sense that no approximation is made along its derivation, constitutes its main feature. The idea encompassed is threefold: application of the Laplace transform to the set of ordinary equations resulting from the approximation, analytical solution of the resulting linear system depending on the complex parameter s and inversion of the transformed angular flux by the Heaviside expansion technique.

We remark that the second step was accomplished by the application of the procedures that we shall describe further ahead. For the *LTSN* approach, exploiting the structure of the corresponding matrix, the inversion was performed by employing the definition of matrix inversion. On the other hand, for the remaining approaches, the matrix inversion was performed by the *Trzaska's* method (Trzaska, Z., 1987).

For the multidimensional transport problems, one of the effective methods to treat linear transport equation is the spectral method (W. Greenberg et al, 1987; M. Mokhtar Kharroubi., 1997; G. Milton Wing., 1962) etc..., whose basic goals is to find exact solution for approximations of the transport equation, several approaches have been suggested. The series expansions method has been largely used in the solution of the differential equation (Kadem, A., 2006). Special functions (Olver, F. W. J., 1974) in particular, Legendre polynomials (Duderstadt, J. J.; Martin, W. R., 1975) expansion have been employed to solve the one-dimensional linear transport the Chebyshev polynomials have been employed to solve the two-dimensional linear transport (Asadzadeh, M and Kadem, A., 2006) and for three dimensional case (Kadem, A., 2007, 2006). According to Gottlieb (Gottlieb, D. and S. A. Orszag, 1977) spectral method involve representation the solution to a problem as a truncated series of known functions of the independent variables, of course there exist other method to determine the coefficients of expansion, but in regard to that, we should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties. Thereby, the orthogonal functions (Szego, G., 1957) and polynomial series have received considerable attention in dealing with various problem. The main characteristic of this technique is that reduces this problems to those of solving a system of algebraic equations, thus greatly simplifying the problem and making it computational plausible.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the z variable. In this way the actual spatial domain may be assumed to be a cylinder with the cross-section Ω and the axial symmetry in z. Then D will correspond to the projection of the points on the unit sphere (the "*speed*") onto the unit disc (which coincides with D,) (Asadzadeh, M., 1986) for the details.

In the present paper we describe an new approximation for the one-dimensional transport equation, using Chebyshev polynomials combined with the Sumudu transform. The approach is based on expansion of the angular flux in a truncated series of Chebyshev polynomials in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique (Belgacem, F. et al, 2003; G. K. Watugala., 1988).

The inversion of the transformed coefficients is obtained using *Trzaska's method* (Trzaska, Z., 1987) and the Heaviside expansion technique. In our knowledge, the combination of the Chebyshev polynomials and the Sumudu Transform to solve the one-dimensional transport equation, in this setting, is not considered in the literature.

2- Analysis

Let us consider the following mono-energetic 3-D transport equation:

$$\underline{\Omega}\,\underline{\nabla}(\underline{r},\underline{\Omega}) + \sigma_t\Psi(\underline{r},\underline{\Omega}) = \int_{4\pi}\sigma_s(\underline{\Omega}.\underline{\Omega}')\Psi(\underline{r},\underline{\Omega}')d\Omega' + \frac{1}{4\pi}Q(\underline{r}) \quad (2.1)$$

where

$$\underline{r} = (x,y,z) = \text{(spatial variable)}. \quad (2.2)$$

$$\underline{\Omega} = (\eta,\xi) = \text{(angular variable)}. \quad (2.3)$$

and

$$\sigma_s(\mu_0) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\sigma_{sk}P_k(\mu_0) \text{ (differential scattering cross section)}. \quad (2.4)$$

with $\mu_0 = \underline{\Omega}.\underline{\Omega}'$ and $P_k =$ the k^{th} Legendre polynomial.

3- Planar Geometry :

We consider a planar-geometry problem with spatial variation only in the x direction:

$$Q(\underline{r}) = q(x), \quad (3.1)$$

$$\Psi(\underline{r},\underline{\Omega}) = \frac{1}{2\pi}\Psi(x,\mu) \quad (3.2)$$

Eq. (2.1) simplifies to

$$\mu\frac{\partial\Psi}{\partial x}(x,\mu) + \sigma_t\Psi(x,\mu) = \int_{-1}^{1}\sigma_s(\mu,\mu')\Psi(x,\mu')d\mu' + \frac{q(x)}{2}, \quad (3.3)$$

with

$$\sigma_s(\mu,\mu') = \sum_{k=0}^{\infty}\frac{2k+1}{2}\sigma_{sk}P_k(\mu)P_k(\mu'). \quad (3.4)$$

So we consider Eq. (3.3) with $0 \le x \le a$ and $-1 \le \mu \le 1$, and subject to the boundary conditions

$$\Psi(x=0,-\mu) = f(\mu). \quad (3.5)$$

and

$$\Psi(x=a,\mu) = 0. \quad (3.6)$$

where $f(\mu)$ is the prescribed incident flux at $x = 0$; $\Psi(x,\mu)$ is the angular flux in the μ direction; σ_t is the total cross section; σ_{sl} with $l = 0, 1, \ldots, L$ are the components of the differential scattering cross section, and $P_k(\mu)$ are the Legendre polynomials of degree *k*.

Theorem 3.1. *Consider the integro-differential equation (3.3) subject to the boundary conditions (3.5) and (3.6), then the function* $\Psi(x,\mu)$ *satisfies the following first-order linear differential equation system for the spatial component* $g_n(x)$

$$\sum_{n=0}^{N} \alpha^1_{n,m} g'_n(x) + \frac{\sigma_t \pi}{2-\delta_{m,0}} g_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} g_n(x) + \frac{q(x)}{2}$$

where

$$\alpha^1_{n,m} := \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu.$$

$$\alpha^2_{n,l} := \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu,$$

$$\alpha^3_{n,l} := \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu,$$

and $g_m(x)$ *are the coefficients of the expansion of the* $\Psi(x,\mu)$.

To prepare the proof of the Theorem (3.1), we need the following result

Proposition 3.2.

Let

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$$

and

$$P_{l-1}(x) = 2xP_l(x) - P_{l-1}(x) - [xP_l(x) - P_{l-1}(x)]/(l+1)$$

then, the recurrence relations for the Chebyshev and the Legendre polynomials respectively we have for $l > 2$ *and k=2,3, take the form*

$$\alpha^k_{n,l+1} := \frac{2l+1}{2l+2}\left[\alpha^k_{n+1,l} + \alpha^k_{n-1,l}\right] - \frac{l}{l+1}\alpha^k_{n,j-1}$$

and, in particular for l=0 and 1, the coefficients $\alpha^2_{n,l}$ *and* $\alpha^3_{n,l}$ *take the values*

$$\alpha^2_{n,l} = \begin{cases} 0 & \text{if } n+l \text{ odd}, \\ \frac{2}{(1+l)^2 - n^2} & \text{if } n+l \text{ even}, \end{cases}$$

and

$$\alpha^3_{n,l} = \frac{\pi \delta_{n,l}}{2 - \delta_{l,0}}$$

Proof

It easy to see that

$$\alpha^1_{n,m} = \frac{\pi \delta_{|n-m|}}{2(2-\delta_{n+m,1})}$$

For *k=2* by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2}[P_l(\mu)T_{n-1}(\mu) - P_l(\mu)T_{n-1}(\mu)] - \frac{l}{2\mu(l+1)}P_{l-1}(\mu)[T_{n+1}(\mu) - T_{n-1}(\mu)]$$

it is known that

$$T_{n+1}(\mu) + T_{n-1}(\mu) = 2\mu T_n(\mu)$$

after doing some algebraic manipulations and integrating over $\mu \in [-1,1]$ on the resulting equation we get

$$\alpha^2_{n,l+1} = \frac{2l-1}{2l+2}\left[\alpha^2_{n+1,l} - \alpha^2_{n-1,l}\right] - \frac{l}{l+1}\alpha^2_{n,j-1}$$

The case *k=3* is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1,1]$ we get the desired result.

Proof of Theorem 3.1

Expanding the angular flux in the μ variable in terms of the Chebyshev polynomials [21] leads to

$$\Psi(x,\mu) = \sum_{n=0}^{N} \frac{g_n(x) T_n(\mu)}{\sqrt{1-\mu^2}} \quad (3.7)$$

with *N=0,2,4,...,* where the expansions coefficients $g_n(x)$ should be determined.

Here $T_n(\mu)$ are the well known Chebyshev polynomials of order *n* which are orthogonal in the interval [-1,1] with respect to the weight $w(t) = 1/\sqrt{1-t^2}$

After replacing Eq. (3.7) into Eq. (3.3) it turns out

$$\sum_{n=0}^{N} \{\mu g'_n(x) + \sigma_t g_n(x)\} \frac{T_n(\mu)}{\sqrt{1-\mu^2}} =$$

$$\sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} P_l(\mu) \sum_{n=0}^{N} g_n(x) \int_{-1}^{1} P_l(\mu') \frac{T_n(\mu')}{\sqrt{1-\mu'^2}} d\mu' + \frac{q(x)}{2} \quad (3.8)$$

using the orthogonality of the Chebyshev polynomials, multiply the Eq. (3.8) by $T_m(\mu)$, considering *m=0,1,...,N,* and integrated in the μ variable in the interval [-1,1]. Thus we get the following first-order linear differential equation system for the spatial component $g_n(x)$.

$$\sum_{n=0}^{N} \alpha^1_{n,m} g'_n(x) + \frac{\sigma_t \pi}{2-\delta_{m,0}} g_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} g_n(x) + \frac{q(x)}{2} \quad (3.9)$$

where

$$\alpha^1_{n,m} = \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu. \quad (3.10)$$

$$\alpha^2_{n,l} = \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu, \quad (3.11)$$

$$\alpha^3_{n,l} = \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu. \quad (3.12)$$

with $\delta_{n,m}$ denoting the delta of Kronecker. Here the coefficients $\alpha^2_{n,l}$ and $\alpha^3_{n,l}$ are evaluated by the multiplication of the Chebyshev and Legendre recurrence formulas and integration of the resulting equation (See proposition 3.2).

In the next step we solve the first-order linear differential equation system (3.9), for this we rewrite this equation in the matrix form

$$A.\frac{dg}{dx}(x) + Bg(x) = C(x) \quad (3.13)$$

where

$g(x) = \text{Col.}\ [g_0(x), g_1(x), \dots, g_N(x)]$ and A and B are squared matrices of order $N+1$ with the components

$$(A)_{i,j} = \alpha^1_{i-1,j-1}. \quad (3.14)$$

$$(B)_{i,j} = \frac{\pi\sigma_t}{2-\delta_{1,j}}\delta_{i,j} - \sum_{l=0}^{L} \frac{2l+1}{2}\sigma_{sl}\alpha^2_{i-1,l} \sum_{n=0}^{N} \alpha^3_{j-1,l} \quad (3.15)$$

and

$$C(x) = \frac{q(x)}{2} = \text{Col.}\ [C_0(x), C_1(x), \dots, C_N(x)]. \quad (3.16)$$

we notice that this equation has the well known solution (Shilling, R.J. et al, 1988).

$$g(x) = e^{-A^{-1}Bx} g(0) + \int_0^x e^{-A^{-1}B(x-\xi)} C(\xi) d\xi \quad (3.17)$$

that depends on vector $g(0)$. Having established an analytical formulation for the exponential appearing in equation (3.17), the $N+1$ unknown components of vector $g(0)$ for the boundary problem (3.3) can be readily obtained applying the boundary conditions (3.5) and (3.6) in the solution given by Eq. (3.7) and multiplying this expression by the Chebyshev polynomial $T_m(\mu)$ considering $m=0,2,4,\dots,$ and integrating in the interval [-1,1], this procedure gives

$$\sum_{n=0}^{N} g_n(0) \int_{-1}^{1} \frac{T_n(\mu) T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = \int_{-1}^{1} g(\mu) T_m(\mu) d\mu \quad (3.18)$$

and

$$\sum_{n=0}^{N} (-1)^n g_n(a) \int_{-1}^{1} \frac{T_n(\mu) T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = 0. \quad (3.19)$$

To derive an analytical formulation for the exponential of matrix $A^{-1}B$, appearing in equation (3.17), let us solve the homogeneous version of equation (3.13), namely

$$A.\frac{dg}{dx}(x) + Bg(x) = 0 \quad (3.20)$$

Now, following the idea of applying the Sumudu transform (cf. Belgacem, F. *et al*, 2003, Theorem 2.2 p. 107) to equation (3.20), we obtain an algebraic linear system that has the solution

$$G(u)\,[uB + A] = R \quad (3.21)$$

with

$R = A.g(0).$

where $G(u) = S\,[g(x)]$ denotes the Sumudu transform of the vector $g(x)$. Solving equation (3.21) that has the solution

$$G(u) = [uB + A]^{-1} R \quad (3.22)$$

by *Trzaska's* method (Trzaska, Z, 1987) the inverse of matrix $[uB - A]$ is readily obtained indeed

$$[uB + A]^{-1} = \sum_{k=1}^{M} \frac{1}{u - s_k} P_k \quad (3.23)$$

where the coefficients s_k denote the eigenvalues of matrix $B^{-1}A$ and the matrices P_k are the ones resulting from the application of *Trzaska's method.* The inversion of the transformed vector $G(u)$ is executed by the Heaviside expansion technique. Following this procedure, we obtain an analytical expression for the exponential of matrix $B^{-1}A$(Trzaska, Z, 1987).

$$e^{-B^{-1}Ax} = \sum_{k=1}^{M} P_k e^{s_k x}. \quad (3.24)$$

We substitute Eq. (3.24) into Eq. (3.17) then, the transformed vector $g(x)$ by the Heaviside technique to get

$$g(x) = \sum_{k=1}^{M} e^{s_k x} P_k R + \sum_{k=1}^{M} P_k \int_0^x e^{s_k(x-\xi)} C(\xi) d\xi. \quad (3.25)$$

Replacing $g_n(0)$ and $g_n(a)$ by its values given by equation (3.17) in equation (3.18) and (3.19), it turns out

$$\sum_{l=1}^{N} \left[\sum_{k=1}^{M} P_k R_l + \sum_{k=1}^{M} P_k \int_0^x e^{A^{-1}B\xi} C(\xi) d\xi \right] \int_{-1}^{1} \frac{T_n(\mu) T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = \int_{-1}^{1} g(\mu) T_m(\mu) d\mu \quad (3.26)$$

and

$$\sum_{l=1}^{N} \left[\sum_{k=1}^{M} P_k e^{s_k a} R_l + \sum_{k=1}^{M} P_k \int_0^x e^{-A^{-1}B(s_k-\xi)} C(\xi) d\xi \right] \int_{-1}^{1} \frac{T_n(\mu) T_m(\mu)}{\sqrt{1-\mu^2}} d\mu = 0 \quad (3.27)$$

with $m=0,2,4,\dots,$ where R_l design the element of the unknown vector R

After solving the linear system (3.26), (3.27) for the components of the vector R the angular flux given by equation (3.7) is completely determined.

Conclusion

The Chebyshev spectral method combined with Sumudu transform should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper, although we have not investigated this idea thoroughly. We will be considering more complicated geometries in future studies, during which we will ascertain this method's usefulness for larger spatial dimensional problems. In preparation for these problems, we are currently investigating the effectiveness of spectral methods combined with Sumudu transform in solving the linear system of differential equation analytically.

We have intention to prove the convergence of the spectral solution within the framework of the analytical solution in our future study by using the discrete ordinates method, combined with the methods employing Sumudu transforms, and we expect this method represent very interesting new ideas for studying the convergence of many numerical methods and can be extended easily to general linear transport problems. In fact only some preliminary results have been obtained. On the other hand in this context we intend to study the existence and uniqueness of its solution by using C_0 *semigroup approach*. Our attention will be focused in this direction.

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