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Moments of Order Statistics from Rayleigh Distribution in the Presence of Outlier Observations

M. E. Moshref[1] and K. S. Sultan[2]

[1] *Department of Mathematics*
Faculty of Science, Al-Azhar University
Nasr City, Cairo 11884, Egypt
E-mail: mmoshref@hotmail.com

[2] *Department of Statistics & Operations Research*
College of Science, King Saud University
P. O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: ksultan@ksu.edu.sa

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Abstract. In this paper, we derive the single and product moments of order statistics from Rayleigh distribution under the contaminations. We assume $X_1,\dots,X_{n-p}$, $p=0,1,2,\dots n$ are independent with probability density function $f(x)$ while $X_{n-p+1},\dots,X_n$ are independent (and independent with $X_1,\dots,X_{n-p}$) and arise from some modified version of $f(x)$ which call $g(x)$ in which the location and/or scale parameters have shifted in value. In addition, we give some numerical illustrations. Finally, some special cases are deduced.

Introduction

Barnett and Lewis (1994) have defined an outlier in a set of data to be "*an observation*" or subset of observations "*which appears to be inconsistent with the remainder of the set of data*". They also describe several models for outlier; two of them are inherent alternative and contamination model. In the first type" inherent alternative" one considers the possibility that the entire set of observations actually comes from a distribution that is different than the originally anticipated. The second type of outlier models is contaminated model. Under this alternative model, one considers the possibility that some of observations come from altered form of the originally anticipated distribution.

It worthwhile to mentioned that for the multiple outlier model, the problem of finding means, variance and covariance of all order statistics will be involve density functions considerably are complicated. Density functions and joint density functions of order statistics arising from a sample of a single outlier have been given by Shu (1978) and David and Shu (1978). One may also refer to Vughan and Venables (1972) for more general expressions of distributions of order statistics using permanent expressions.

Arnold and Balakrishnan (1989) have obtained the density function of $X_{r:n}$ when the sample of size n contains unidentified single outlier. They also obtained the joint density function of $X_{r:n}$ and $X_{s:n}$, $1\le r<s\le n$. Balakrishnan (1994b) has derived some recurrence relations satisfied by the single product moments of order statistics from the right truncated exponential distribution. Also he has deduced the recurrence relations for the multiple outlier models (with slippage of observations), see also Balakrishnan (1994a). Childs, Balakrishnan and Moshref (2001) have derived some recurrence relations for the single and product moments of order

statistics from n independent and non-identically distributed Lomax and the right-truncated Lomax random variables. Barnett and Lewis (1994) say that "*... we are not aware of any published application to studies of robustness of accommodation procedures in the presence of multiple outliers. There is much work waiting to be done in this important area*".

Let $X_1, \ldots, X_{n-p}$ are independent with probability density function $f(x)$ and cumulative distribution function $F(x)$ while $X_{n-p+1}, \ldots, X_n$ are independent (and independent with $X_1, \ldots, X_{n-p}$) with probability density function $g(x)$ and cumulative distribution function $G(x)$. Let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics obtained by arranging $X_1, \ldots, X_n$ in increasing order magnitude. The probability density function of the r^{th} order statistics $X_{r:n}$ under the multiple outlier model can be written as follows: [see Childs (1996)]

$$\begin{aligned} f_{r:n}[p](x) = & \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 f(x)[F(x)]^s \\ & \times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s-1} \\ & \times [1-G(x)]^{p-r+s+1} \\ & + \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 g(x)[F(x)]^s \\ & \times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s} \\ & \times [1-G(x)]^{p-r+s}, \\ & 1 \le r \le n,\ p = 0,1,2,\ldots,n,\ -\infty < x < \infty, \end{aligned} \tag{1.1}$$

where $C_1 = \dfrac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)!}$,

and

$$C_2 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)!}.$$

Similarly, the joint density function of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is given by

$$\begin{aligned} f_{r,s:n}[p](x,y) = & \sum_{j=0}^{s-r-1} \sum_{s=\max(0,s-p-j-2)}^{\min(n-p-j-2,r-1)} A_1 f(x) f(y)[F(x)]^i \\ & \times [G(x)]^{r-1-i}[F(y)-F(x)]^j \\ & \times [G(y)-G(x)]^{s-r-i-j}[1-F(y)]^{n-p-i-j-2} \\ & \times [1-G(y)]^{p-s+i+j+2} \\ & + \sum_{j=0}^{s-r-1} \sum_{s=\max(0,s-p-j-2)}^{\min(n-p-j-1,r-1)} A_2 f(x) g(y)[F(x)]^i \\ & \times [G(x)]^{r-1-i}[F(y)-F(x)]^j \\ & \times [G(y)-G(x)]^{s-r-1-j}[1-F(y)]^{n-p-i-j-1} \\ & \times [1-G(y)]^{p-s+i+j+1} \\ & + \sum_{j=0}^{s-r-1} \sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)} A_2 g(x) f(y)[F(x)]^i \\ & \times [G(x)]^{r-1-i}[F(y)-F(x)]^j \\ & \times [G(y)-G(x)]^{s-r-1-j}[1-F(y)]^{n-p-i-j-1} \\ & \times [1-G(y)]^{p-s+i+j+1} \\ & + \sum_{j=0}^{s-r-1} \sum_{i=\max(0,s-p-j)}^{\min(n-p-j,r-1)} A_3 g(x) g(y)[F(x)]^i \\ & \times [G(x)]^{r-1-i}[F(y)-F(x)]^j \\ & \times [G(y)-G(x)]^{s-r-1-j}[1-F(y)]^{n-p-i-j} \\ & \times [1-G(y)]^{p-s+i+j},\ p = 0,1,2,\ldots n, \\ & -\infty < x < y < \infty,\ 1 \le r < s < n, \end{aligned} \tag{1.2}$$

where

$$A_1 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j-2)!(p-s+i+j+2)!},$$

$$A_2 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j-1)!(p-s+i+j+1)!},$$

and

$$A_3 = \frac{(n-p)!p!}{i!(r-i-1)!j!(s-r-1-j)!(n-p-i-j)!(p-s+i+j)!}.$$

Setting $p = 1$ in (1.1) and (1.2), we obtain the corresponding pdf's in the case of the single outlier given by Shu (1978) and David and Shu (1978).

In this paper, we consider the case when the variables $X_1, \ldots, X_{n-p}$ are independent observation from Rayleigh distribution with density

$$f(x)=\frac{x}{\theta^2}\exp[-\frac{x^2}{2\theta^2}],\ x\geq 0,\ \theta>0,$$

and $X_{n-p+1},\dots,X_n$ arise from the same distribution with density

$$g(x)=\frac{x}{\tau^2}\exp[-\frac{x^2}{2\tau^2}],\ x\geq 0,\ \tau>0,$$

where $\tau>\theta$.

The corresponding cumulative distribution functions $F(x)$ and $G(x)$ are given as

$$F(x)=1-\exp[-\frac{x^2}{2\theta^2}],\ x\geq 0,\ \theta>0,$$

and

$$G(x)=1-\exp[-\frac{x^2}{2\tau^2}],\ x\geq 0,\ \tau>0.$$

The relation between $f(x)$ and $F(x)$ is given by

$$f(x)=\frac{x}{\theta^2}[1-F(x)]. \qquad (1.3)$$

Similarly

$$g(x)=\frac{x}{\tau^2}[1-G(x)]. \qquad (1.4)$$

In the next two sections, we use (1.3) and (1.4) to derive the single and product moments of order statistics from Rayleigh distribution under the multiple outlier model. This situation is known as a multiple outlier model with slippage of p observations; see Barnett and Lewise (1994). This specific multiple outlier model was introduced by David (1979).

Single moments

In this section we derive the k^{th} moment of the r^{th} order statistic under multiple outlier model (with a slippage of p observations). Let $\mu_{r:n}^{(k)}[p],\ 1\leq r\leq n$ denote the k^{th} single moment of order statistics in the presence of $p-$outlier observations from Rayleigh distribution. The following theorem gives an explicit form of $\mu_{r:n}^{(k)}[p]$.

Theorem 1

For $1\leq r\leq n,\ p=0,1,\dots,n$ and $k=0,1,\dots$ the single moments $\mu_{r:n}^{(k)}[p]$ is given by

$$\mu_{r:n}^{(k)}[p]=\frac{\Gamma(\frac{k}{2}+1)}{2^{-k/2}}\Bigg(\frac{1}{\theta^2}\sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)}C_1\times\sum_{\ell=0}^{s}\binom{s}{\ell}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}\times\frac{-1^{\ell+m}}{(\frac{n-p-s+\ell}{\theta^2}+\frac{p-r+s+1+m}{\tau^2})^{\frac{k}{2}+1}}$$

$$+\frac{1}{\tau^2}\sum_{s=\max(0,r-p)}^{\min(n-p,r-1)}C_2\times\sum_{\ell=0}^{s}\binom{s}{\ell}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}\times\frac{-1^{\ell+m}}{(\frac{n-p-s+\ell}{\theta^2}+\frac{p-r+s+1+m}{\tau^2})^{\frac{k}{2}+1}}\Bigg), \qquad (2.1)$$

where $C_1=\frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)!}$,

and

$$C_2=\frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)!}.$$

Proof

Starting from (1.1), we have

$$
\begin{aligned}
\mu_{r:n}^{(k)}[p] &= \int_0^{\infty} x^k f_{r:n}[p]dx \\
&= \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 \int_0^{\infty} x^k f(x)[f(x)]^s \\
&\times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s-1} \\
&\times [1-G(x)]^{p-r+s+1}dx \\
&+ \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 \int_0^{\infty} x^k g(x)[F(x)]^s \\
&\times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s} \\
&\times [1-G(x)]^{p-r+s+1}dx \\
&= \frac{1}{\theta^2} \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 \int_0^{\infty} x^{k+1}[f(x)]^s \\
&\times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s} \\
&\times [1-G(x)]^{p-r+s+1}dx \\
&+ \frac{1}{\tau^2} \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 \int_0^{\infty} x^{k+1}[f(x)]^s \\
&\times [G(x)]^{r-s-1}[1-F(x)]^{n-p-s} \qquad (2.2) \\
&\times [1-G(x)]^{p-r+s+1}dx .
\end{aligned}
$$

By using the differential equation (1.3) and (1.4) in (2.2), we have

$$
\begin{aligned}
\mu_{r:n}^{(k)}[p] &= \frac{1}{\theta^2} \sum_{s=\max(0,r-p-1)}^{\min(n-p-1,r-1)} C_1 \int_0^{\infty} x^{k+1} \sum_{\ell=0}^{s} \binom{s}{\ell} \\
&\times \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{\ell+m} [1-F(x)]^{n-p-s-\ell} \\
&\times [1-G(x)]^{p-r+s+m+1}dx \\
&= \frac{1}{\tau^2} \sum_{s=\max(0,r-p)}^{\min(n-p,r-1)} C_2 \int_0^{\infty} x^{k+1} \sum_{\ell=0}^{s} \binom{s}{\ell} \qquad (2.3) \\
&\times \sum_{m=0}^{r-s-1} \binom{r-s-1}{m} (-1)^{\ell+m} [1-F(x)]^{n-p-s+\ell} \\
&\times [1-G(x)]^{p-r+s+1+m} dx .
\end{aligned}
$$

It is easy to derive (2.1) by writing $F(x)=1-(1-F(x))$ and $G(x)=1-(1-G(x))$ in (2.3) and integrate over x . Table (1) displays the numerical values of $\mu_{r:n}^{(k)}[p]$ for some values of n,p,r,θ,τ.

Product moments

In this section, we derive the product moments of order statistics under multiple outlier model (with a slippage of p observations).

Let $\mu_{r,s,n}^{(k,m)}[p]$, $(1 \le r < s \le n)$ denote the (k^{th}, m^{th}) product moments of the order statistics (r^{th}, s^{th}) order statistics in the presence of the $p-$ outlier observation from Rayleigh distribution. The following theorem gives an explicit form of $\mu_{r,s,n}^{(k,m)}[p]$.

Theorem 2

For $1 \le r < s \le n$, $p = 0,1,2,...,n$ and $k,m = 0,1,2,...$, the (k^{th}, m^{th}) product moments of the (r^{th}, s^{th}) order statistics in the presence of $p-$ outlier observations from Rayleigh distribution is given by

$$
\begin{aligned}
\mu_{r,s:n}^{(k,m)}[p] &= \frac{\Gamma(\frac{m+k}{2}+2)}{2\theta^4} \sum_{j=0}^{s-r-1} \sum_{i=\max(0,s-p-j-2)}^{\min(n-p-j-2,r-1)} A_1 \sum_{b=0}^{i} \binom{i}{b} \\
&\times \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q} \\
&\times (-1)^{b+d+\ell+q} \frac{I_{\frac{z_1}{z_1+z_2}}\left(\frac{k}{2}+1, \frac{m}{2}+1\right)}{z_1^{\frac{k}{2}+1} z_2^{\frac{m}{2}+1}} \qquad (\\
&+ \frac{1}{2\theta^2\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)} A_2 \sum_{b=0}^{i} \binom{i}{b} \\
&\times \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q}
\end{aligned}
$$

$$\times(-1)^{b+d+\ell+q}\frac{I_{\frac{z_1}{z_1+z_2}}\left(\frac{k}{2}+1,\frac{m}{2}+1\right)}{z_1^{\frac{k}{2}+1}z_2^{\frac{m}{2}+1}}$$

$$+\frac{\Gamma(\frac{m+k}{2}+2)}{2\theta^2\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2\sum_{b=0}^{i}\binom{i}{b}$$

$$\times\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{\ell=0}^{j}\binom{j}{\ell}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}$$

$$\times(-1)^{b+d+\ell+q}\frac{I_{\frac{z_3}{z_3+z_4}}\left(\frac{k}{2}+1,\frac{m}{2}+1\right)}{4z_3^{\frac{k}{2}+1}z_4^{\frac{m}{2}+1}}$$

$$+\frac{1}{2\tau^4}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j)}^{\min(n-p-j,r-1)}A_3\sum_{b=0}^{i}\binom{i}{b}$$

$$\times\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{\ell=0}^{j}\binom{j}{\ell}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}$$

$$\times(-1)^{b+d+\ell+q}\frac{I_{\frac{z_3}{z_3+z_4}}\left(\frac{k}{2}+1,\frac{m}{2}+1\right)}{z_3^{\frac{k}{2}+1}z_4^{\frac{m}{2}+1}},\tag{3.1}$$

where

$$z_1=\frac{1}{2}\left\{\frac{b+j-\ell+1}{\theta^2}+\frac{d+s-r-1-j-q}{\tau^2}\right\},\tag{3.2}$$

$$z_2=\frac{1}{2}\left\{\frac{n-p-i-j-1+\ell}{\theta^2}+\frac{p-s+r+j+2+q}{\tau^2}\right\},\tag{3.3}$$

$$z_3=\frac{1}{2}\left\{\frac{b+j-\ell}{\theta^2}+\frac{d+s-r-j-q}{\tau^2}\right\},\tag{3.4}$$

$$z_4=\frac{1}{2}\left\{\frac{n-p-i-j+\ell}{\theta^2}+\frac{p-s+i+j+q}{\tau^2}\right\},\tag{3.5}$$

and

$$I_a\left(\frac{k}{2}+1,\frac{m}{2}+1\right)=\int_0^a u^{\frac{k}{2}}(1-u)^{\frac{m}{2}}du.\tag{3.6}$$

Proof

Starting from (1.2), we have

$$\mu_{r,s:n}^{(k,m)}[p]=\int_0^\infty\int_0^y x^k y^m f_{r,s:n}[p](x,y)dxdy$$

$$=\frac{1}{\theta^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-2)}^{\min(n-p-j-2,r-1)}A_1$$

$$\times\int_0^\infty\int_0^y x^k y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$

$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$

$$\times[1-F(y)]^{n-p-i-j-1}[1-G(y)]^{p-s+i+j+2}$$

$$\times f(x)dxdy$$

$$+\frac{1}{\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2$$

$$\times\int_0^\infty\int_0^y x^k y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$

$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$

$$\times[1-F(y)]^{n-p-i-j-1}[1-G(y)]^{p-s+i+j+2}$$

$$\times f(x)dxdy$$

$$+\frac{1}{\theta^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2$$

$$\times\int_0^\infty\int_0^y x^k y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$

$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$

$$\times[1-F(y)]^{n-p-i-j}[1-G(y)]^{p-s+i+j+1}$$

$$\times g(x)dxdy$$

$$+\frac{1}{\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j)}^{\min(n-p-j-1,r-1)}A_3$$

$$\times\int_0^\infty\int_0^y x^k y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$

$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$

$$\times[1-F(y)]^{n-p-i-j}[1-G(y)]^{p-s+i+j+1}$$

$$\times g(x)dxdy$$

$$=\frac{1}{\theta^4}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_1$$
$$\times\int_0^\infty\int_0^y x^{k+1}y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$
$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$
$$\times[1-F(y)]^{n-p-i-j-1}[1-G(y)]^{p-s+i+j+2}$$
$$\times[1-F(x)]dxdy$$

$$+\frac{1}{\theta^2\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2$$
$$\times\int_0^\infty\int_0^y x^{k+1}y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$
$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$
$$\times[1-F(y)]^{n-p-i-j-1}[1-G(y)]^{p-s+i+j+2}$$
$$\times[1-F(x)]dxdy$$

$$+\frac{1}{\theta^2\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2$$
$$\times\int_0^\infty\int_0^y x^{k+1}y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$
$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$
$$\times[1-F(y)]^{n-p-i-j}[1-G(y)]^{p-s+i+j+1}$$
$$\times[1-G(x)]dxdy$$

$$+\frac{1}{\tau^4}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j)}^{\min(n-p-j,r-1)}A_3$$
$$\times\int_0^\infty\int_0^y x^{k+1}y^{m+1}[F(x)]^i[G(x)]^{r-1-i}$$
$$\times[F(y)-F(x)]^j[G(y)-G(x)]^{s-r-1-j}$$
$$\times[1-F(y)]^{n-p-i-j}[1-G(y)]^{p-s+i+j+1}$$
$$\times[1-G(x)]dxdy. \quad (3.7)$$

By using (1.3) and (1.4) in (3.7), we have

$$\mu_{r,s:n}^{(k,m)}[p]=\frac{1}{\theta^4}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-2)}^{\min(n-p-j-2,r-1)}A_1\sum_{b=0}^{i}\binom{i}{b}$$
$$\times\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{\ell=0}^{j}\binom{j}{\ell}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}$$
$$\times(-1)^{b+d+\ell+q}\int_0^\infty\int_0^y x^{k+1}y^{m+1}[1-F(x)]^{b+j-\ell+1}$$
$$\times[1-G(x)]^{d+s-r-j}[1-F(y)]^{n-p-i-j+\ell}$$
$$\times[1-G(y)]^{p-s+i+j+2+q}dxdy$$
$$+\frac{1}{\theta^2\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=\max(0,s-p-j-1)}^{\min(n-p-j-1,r-1)}A_2\sum_{b=0}^{i}\binom{i}{b}$$
$$\times\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{\ell=0}^{j}\binom{j}{\ell}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}$$
$$\times(-1)^{b+d+\ell+q}\int_0^\infty\int_0^y x^{k+1}y^{m+1}[1-F(x)]^{p+j-\ell}$$
$$\times[1-G(x)]^{d+s-r-j-q}[1-F(y)]^{n-p-i-j+\ell}$$
$$\times[1-G(y)]^{p-s+i+j+1+q}dxdy$$
$$+\frac{1}{\tau^4}\sum_{i=\max(0,s-p-j)}^{\min(n-p-j,r-1)}A_3\sum_{b=0}^{i}\binom{i}{b}$$
$$\times\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{\ell=0}^{j}\binom{j}{\ell}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}$$
$$\times(-1)^{b+d+\ell+q}\int_0^\infty\int_0^y x^{k+1}y^{m+1}[1-F(x)]^{p+j-\ell}.$$
$$\times[1-G(x)]^{d+s-r-j-q}[1-F(y)]^{n-p-i-j+\ell}$$
$$\times[1-G(y)]^{p-s+i+j+1+q}dxdy. \quad (3.8)$$

Upon, we put $F(x)=1-[1-F(x)]$, $G(x)=1-[1-G(x)]$, $F(y)-F(x)=[1-F(x)]-[1-F(y)]$ and $G(y)-G(x)=[1-G(x)]-[1-G(y)]$ and using binomial theorem, where

$1-F(x)=\exp[-\frac{x^2}{2\theta^2}]$, $1-F(y)=\exp[-\frac{y^2}{2\theta^2}]$

and

$1-G(x)=\exp[-\frac{x^2}{2\tau^2}]$, $1-G(y)=\exp[-\frac{y^2}{2\tau^2}]$, we get (3.1).

Table 2 given below displays the product moments and the corresponding covariance of order statistics in (3.1) when $\theta=1$, $\tau=1/4$ and $p=0,1,2$.

Special Cases

In this section, we deduce some special cases from the single and product moments given in (2.1) and (2.2) as follows:

1. Setting $p=0$, we get the single and product moments of order statistics when $x_1,\ldots,x_n$ have Rayleigh distribution with parameters θ, see Dyer and Whisenand (1973)

$$\mu_{r:n}^{(k)}[0]=\frac{n!\Gamma(k/2+1)}{(r-1)!(n-r)!2^{-k/2}}\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\times(-1)^{\ell}\frac{\theta^k}{(n-r+1+\ell)^{k/2+1}}, \quad (4.1)$$

And

$$\mu_{r,s:n}^{(k,m)}[0]=\frac{n!\Gamma((k+m)/2+2)}{(r-1)!(s-r-1)!(n-r)!2\theta^4}\times\sum_{b=0}^{r-1}\binom{r-1}{b}\sum_{\ell=0}^{s-r-1}\binom{s-r-1}{\ell}\times(-1)^{b+\ell}\frac{I_{\frac{z_1}{z_1+z_2}}(k/2+1,\, m/2+1)}{z_1^{-(k/2+1)}z_2^{-(m/2+1)}}, \quad (4.2)$$

Where

$$z_1=\frac{b+s-r-\ell}{2\theta^2} \quad \text{and} \quad z_2=\frac{n-s-1-\ell}{2\theta^2}$$

2. If we put $p=n$, we have the same relation above but with parameter τ.
3. if we put $p=1$, we have the relation for the single outlier case.

Table 1. Mean and variance of order statistics when $\theta=1.0$ and $\tau=2.0$

p	r	n	$\mu_{r,:n}^{(1)}[p]$	Variance
0	1	5	0.5605	0.0858
0	2	5	0.8913	0.1056
0	3	5	1.1992	0.1287
0	4	5	1.5481	0.1700
0	5	5	2.0675	0.2920
1	1	5	0.6079	0.1010
1	2	5	0.9757	0.1271
1	3	5	1.3328	0.1625
1	4	5	1.7768	0.2453
1	5	5	2.8266	1.1192
2	1	5	0.6699	0.1226
2	2	5	1.0887	0.1597
2	3	5	1.5211	0.2225
2	4	5	2.1690	0.5019
2	5	5	3.3244	1.2889

Table 2. The product moments of order statistics when $\theta=1.0$ and $\tau=2.0$

p	r	s	n	$\mu_{r,s:n}^{(1,1)}[p]$
0	1	2	5	1.1182
0	1	3	5	1.4360
0	2	3	5	2.3013
0	1	4	5	1.8081
0	2	4	5	2.8898
0	3	4	5	3.9188
0	1	5	5	2.3734
0	2	5	5	3.7857
0	3	5	5	5.1178
0	4	5	5	6.6669
1	1	2	5	1.2775
1	1	3	5	3.9517
1	2	3	5	5.2084
1	1	4	5	4.7535
1	2	4	5	11.0937
1	3	4	5	9.6252
1	1	5	5	5.1046
1	2	5	5	12.8589
1	3	5	5	18.2291
1	4	5	5	15.0532
2	1	2	5	1.5320
2	1	3	5	6.8859
2	2	3	5	9.6910
2	1	4	5	12.3920
2	2	4	5	29.5163
2	3	4	5	24.7458
2	1	5	5	10.0523
2	2	5	5	37.2539
2	3	5	5	50.8036
2	4	5	5	40.4444

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