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Approximating Singular Integral of Cauchy Type with Unbounded Weight Function on the Edges

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Abstract: In this paper discrete vortices method (MDV) is developed for approximating the singular integral (SI) of Cauchy type with unbounded weight function on the edges. Linear interpolation spline and modification discrete vortex method are used in constructing the quadrature formula (QF) for the above mentioned integral. The convergence of quadrature formula are examined in the classes of functions $H^{\alpha}([-1,1], A)$ and $C^{1}([-1,1])$. New algorithms are presented to evaluate the singular integrals with weight function. At the end of the paper numerical examples are given to show the validity of the quadrature formula.

Introduction

The theories of ordinary and partial differential equations are fruitful sources of integral equations. In the quest for the representation formula for the solution of an ordinary and partial differential equations with boundary or initial conditions one is always led to an integral equation of Fredholm, Volterra, Cauchy or Hilbert type (see Kanwal R.P., 1997, Moiseiwitsch B.L. (1977)). Once a boundary value or initial value problem has been formulated in terms of integral equations, it becomes possible to solve this problem using various technique (see Kanwal R.P., 1997.)

In many practical situation (for example see Chen,Y.Z., 2004, Lifanov I.K. and Polonskii I.E. (1975).), one always encounter the problem of solving numerically the singular integral equations of the first kind on the segment of the form

$$\int_{-1}^{1}\frac{k(x,t)\varphi(t)}{x-t}dt+\int_{-1}^{1}\bar{k}(x,t)\varphi(t)dt=f(x), \qquad (1)$$

$$-1<x<1,$$

where $k(x,t)$ and $\bar{k}(x,t)$ are regular squre-integrable functions of two variables t and x with $k(t,t)\neq 0$ and $f(x)\in H(\alpha),\ 0<\alpha\leq 1$. The solutions of equation (1) and the corresponding characteristic equation:

$$\int_{-1}^{1}\frac{k(x,t)}{t-x}\varphi(t)dt=f(x), \quad -1<x<1$$

where $\varphi(x)$ is to be determinant, have the same specific features.

It is known that if a function $\varphi(x)$ satisfies Holder condition then the singular integrals in (2) exist (see Muskhelishvili N.I., 1953). Moreover, when $k(x,t)=1$, and for $\kappa=1$, (in this case, we say that the solution of (2) may be unbounded at both end-points of the segment [-1,1]) solution of the equation (2) (see Lifanov I.K., et al., 2004, p.5) is given by the formula:

$$\varphi(x)=-\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}f(t)dt}{x-t}+\frac{C}{\pi^2\sqrt{1-x^2}}, \quad (3)$$

where C is an arbitrary constant and

$$\int_{-1}^{1}\varphi(x)dx=C.$$

Cauchy singular integral equations occur in a variety of physical problems, especially in connection with the solution of partial differential equations in $\mathbb{R}^2$ or $\mathbb{R}^3$. There are many monographs and papers written on SIE; (see Lifanov I.K., et al., 2004, Belotserkovskii S.M. and Lifanov I.K., 1985, Muskhelishvili N.I., 1953) and the references contained therein.

From the cource of mathematical analysis it follows that the integral on the right side of (3) for different function *f(x)* in many cases is not integrable in closed form. Therefore the approximation method is to be apllied to evaluate the singular integral (3). There are many methods and approaches to evaluate the singular integral at any singular point x (see Palamara A.O. 1990, Smith H.V., 2006).

In this paper we develop MDV to approximate the singular integral in (3) by using the modification of discrete vortices method and linear spline. It enables us to provide the convergence of quadrature formula in different classes of functions for any singular point $x \in (-1,1)$.

Construction of the quadrature formula

We consider the following singular integral

$$I(x) = \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{t-x}dt, \quad -1 < x < 1, \tag{4}$$

Let $t_k = -1 + kh, k = 0,...,(N+1)$ be equal partition of the segment [-1,1] with width $h = 2/(N+1)$, where N is a positive integer number. Let $E = \{t_k, k = 1,...,N\}$ and $Q(j) = \{j-1, j, j+1, \ j = 1,...,N\}$. It is assume that j is fixed number. The set E is called a canonic partition of the interval [-1,1] (see Belotserkovskii, S.M. and Lifanov, I.K., 1985, p.13).

Let $s_\nu(t)$ be a linear interpolation spline of the form (see Stechkin, S and Subbotin U., 1976):

$$s_\nu = \frac{t-t_\nu}{h} f(t_{\nu+1}) + \frac{t_{\nu+1}-t}{h} f(t_\nu) \tag{5}$$

which has the following properties:

a) if $f(t) = c$ then $s_\nu(t) = c$,

b) if $f(t) = at + b$, then $s_\nu(t) = at + b$,

where $t \in [t_\nu, t_{\nu+1}]$.

As singular point x doesn't coincide with knot points that is $x \notin E$ and various in the interval (t_j, t_{j+1}) we can write it as $x = t_j + \varepsilon$, where $\varepsilon \in (0,h)$. Now QF for SI (4), is constructed in the following way

$$\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{t-(t_j+\varepsilon)}dt$$

$$= \frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}} \left[\sum_{k=0,\, k\notin Q(j)}^{N} \int_{t_k}^{t_{k+1}} \frac{\sqrt{1-t^2}f(t)}{t-(t_j+\varepsilon)}dt + \right. \tag{6}$$

$$\left. + \sum_{k=j-1}^{j+1} \int_{t_k}^{t_{k+1}} \frac{\sqrt{1-t^2}f(t)}{t-(t_j+\varepsilon)}dt \right] = \sum_{\substack{k=0,\\ k\notin Q(j)\cup j+2}}^{N+1} A_k(t_j+\varepsilon)f(t_k) +$$

$$+ \frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}} \sum_{\nu=j-1}^{j+1} \int_{t_l}^{t_{l+1}} \frac{\sqrt{1-t^2}s_\nu(t)}{t-(t_j+\varepsilon)}dt = R_N(t_j+\varepsilon),$$

where

$$A_k(t_j+\varepsilon) = \frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}} \frac{\sqrt{1-t_k^2}\,h}{t_k-(t_j+\varepsilon)}, k = 1,2,...,j-2, j+3,...,N. \tag{7}$$

These coefficients $A_k, k = 1,...,j-2, j+3,...,N$ and the coefficients defined by discrete vortex method are similar (Belotserkovskii, S.M. and Lifanov, I.K., 1985,p.19-25). Now the coefficients A_0 and A_{N+1} are found from the condition that the quadrature formula (6) is exact for the linear function f, i.e.:

$$\left.\begin{aligned}
A_0(t_j+\varepsilon) &= \frac{1}{2\pi\sqrt{1-(t_j+\varepsilon)^2}} \left[\int_{-1}^{1} \frac{\sqrt{1-t^2}(1-t)}{t-(t_j+\varepsilon)}dt - \sum_{\nu=j-1}^{j+1} \int_{t_\nu}^{t_{\nu+1}} \frac{\sqrt{1-t^2}(1-t)}{t-(t_j+\varepsilon)}dt - \sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N} \frac{\sqrt{1-t_k^2}(1-t_k)}{t_k-(t_j+\varepsilon)}h \right], \\
A_{N+1}(t_j+\varepsilon) &= \frac{1}{2\pi\sqrt{1-(t_j+\varepsilon)^2}} \left[\int_{-1}^{1} \frac{\sqrt{1-t^2}(1+t)}{t-(t_j+\varepsilon)}dt - \sum_{\nu=j-1}^{j+1} \int_{t_\nu}^{t_{\nu+1}} \frac{\sqrt{1-t^2}(1+t)}{t-(t_j+\varepsilon)}dt - \sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N} \frac{\sqrt{1-t_k^2}(1+t_k)}{t_k-(t_j+\varepsilon)}h \right].
\end{aligned}\right\} \tag{8}$$

Substituting (7) and (8) into (6), we obtain:

$$\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\varphi(t)}{t-(t_j+\varepsilon)}dt$$

$$=\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\left\{\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{\sqrt{1-t_k^2}\varphi(t_k)}{t_k-(t_j+\varepsilon)}h+\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{\sqrt{1-t^2}s_v^*(t)}{t-(t_j+\varepsilon)}dt\right\}+R_N(t_j+\varepsilon),\qquad(9)$$

where

$$\varphi(t)=f(t)-\frac{1}{2}\left[(1-t)f(-1)+(1+t)f(1)\right],\qquad(10)$$

$$s_v^*(t)=s_v(t)-\frac{1}{2}\left[(1-t)f(-1)+(1+t)f(1)\right].\qquad(11)$$

Substitute (10) and (11) into (9) and after simplifying the corresponding terms yields

$$\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}f(t)}{t-(t_j+\varepsilon)}dt$$

$$=\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}A_k(t_j+\varepsilon)\varphi(t_k)+$$

$$+A_0(t_j+\varepsilon)f(-1)+A_{N+1}(t_j+\varepsilon)f(1)+$$

$$+A_{j-1}(t_j+\varepsilon)f(t_{j-1})+A_j(t_j+\varepsilon)f(t_j)+\qquad(12)$$

$$+A_{j+1}(t_j+\varepsilon)f(t_{j+1})+A_{j+2}(t_j+\varepsilon)f(t_{j+2})+$$

$$+R_N(t_j+\varepsilon),$$

where $A_k(t_{0j})$, $k=1,...,j-2,j+3,...,N$ is defined by (7) and the rest of the coefficients are:

$$A_0(t_j+\varepsilon)=\frac{1}{2\pi\sqrt{1-(t_j+\varepsilon)^2}}\left(\int_{-1}^{t_{j-1}}+\int_{t_{j+2}}^{1}\right)\frac{(1-t)\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt,$$

$$A_{N+1}(t_j+\varepsilon)=\frac{1}{2\pi\sqrt{1-(t_j+\varepsilon)^2}}\left(\int_{-1}^{t_{j-1}}+\int_{t_{j+2}}^{1}\right)\frac{(1+t)\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt,$$

$$A_{j-1}(t_j+\varepsilon)=\frac{1}{\pi h\sqrt{1-(t_j+\varepsilon)^2}}\int_{t_{j-1}}^{t_j}\frac{(t_j-t)\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt,$$

$$A_j(t_j+\varepsilon)=\frac{1}{\pi h\sqrt{1-(t_j+\varepsilon)^2}}\left(\int_{t_{j-1}}^{t_j}\frac{(t-t_{j-1})\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt+\int_{t_j}^{t_{j+1}}\frac{(t_{j+1}-t)\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt\right),$$

$$A_{j+1}(t_j+\varepsilon)=\frac{1}{\pi h\sqrt{1-(t_j+\varepsilon)^2}}\left(\int_{t_j}^{t_{j+1}}\frac{(t-t_j)\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt+\int_{t_{j+1}}^{t_{j+2}}\frac{(t-t_{j+1})\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt\right).$$

$$A_{j+2}(t_j+\varepsilon)=\frac{1}{\pi h\sqrt{1-(t_j+\varepsilon)^2}}\int_{t_{j+1}}^{t_{j+2}}\frac{(t-t_{j+1})\sqrt{1-t^2}}{t-(t_j+\varepsilon)}dt.\qquad(13)$$

Let us introduce the following designations

$$P_1(t,x)=\frac{t\sqrt{1-x^2}-x\sqrt{1-t^2}}{\sqrt{1-x^2}+\sqrt{1-t^2}},$$

$$P_2(t)=arcsin(t),\qquad(14)$$

$$P_3(t)=\sqrt{1-t^2},$$

$$P_4(t)=t\sqrt{1-t^2}.$$

From (14) and antiderivatives rule it follows that:

$$\left.\begin{aligned}&\int_{a_1}^{a_2}\frac{dt}{\sqrt{1-t^2}(t-x)}=\frac{1}{\sqrt{(1-x^2)}}\ln\left|\frac{P_1(a_2,x)}{P_1(a_1,x)}\right|,\\&\int_{a_1}^{a_2}\frac{dt}{\sqrt{1-t^2}}=P_2(a_2)-P_2(a_1),\\&\int_{a_1}^{a_2}\frac{tdt}{\sqrt{1-t^2}}=P_3(a_1)-P_2(a_2),\\&\int_{a_1}^{a_2}\frac{t^2dt}{\sqrt{1-t^2}}=\frac{1}{2}\left[P_4(a_1)-P_4(a_2)+P_2(a_2)-P_2(a_1)\right].\end{aligned}\right\}\qquad(15)$$

Now computing the expressions on the right hand side of (13), taking into account (15) and rearranging corresponding terms, we arrive QF for actual evaluation:

$$
\begin{aligned}
&\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f(t)}{t-(t_j+\varepsilon)}dt = \\
&= \sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N} A_k(t_j+\varepsilon)\varphi(t_k) + \\
&+ A_0(t_j+\varepsilon)f(-1) + A_{N+1}(t_j+\varepsilon)f(1) + \\
&+ A_{j-1}(t_j+\varepsilon)f(t_{j-1}) + A_j(t_j+\varepsilon)f(t_j) + \\
&+ A_{j+1}(t_j+\varepsilon)f(t_{j+1}) + A_{j+2}(t_j+\varepsilon)f(t_{j+2}) + \\
&+ R_N\left(t_j+\varepsilon\right),
\end{aligned}
\tag{16}
$$

where the coefficient of QF are defined by the following formulas:

$$
\begin{aligned}
&A_k\left(t_j+\varepsilon\right) = A_k^{(1)}\left(t_j+\varepsilon\right) + \\
&+\frac{\left(t_j+\varepsilon\right)+t_k}{\sqrt{1-\left(t_j+\varepsilon\right)^2}}\frac{h}{\sqrt{1-t_k^2}}, k=1,\dots,j-2,j+3,\dots,N,
\end{aligned}
$$

$$
\begin{aligned}
&A_0\left(t_j+\varepsilon\right) = A_0^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{2\pi\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[\left(t_j+\varepsilon\right)\left(\pi+P_2\left(t_{j-1}\right)-P_2\left(t_{j+2}\right)\right)-\left(1-\left(t_j+\varepsilon\right)\right)\left(P_3\left(t_{j-1}\right)-P_3\left(t_{j+2}\right)\right)- \\
&-\frac{1}{2}\left(P_4\left(t_{j+2}\right)-P_4\left(t_{j-1}\right)+P_2\left(t_{j-1}\right)-P_2\left(t_{j+2}\right)+\pi\right).
\end{aligned}
$$

$$
\begin{aligned}
&A_{N+1}\left(t_j+\varepsilon\right) = A_{N+1}^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{2\pi\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[\left(t_j+\varepsilon\right)\left(\pi+P_2\left(t_{j-1}\right)-P_2\left(t_{j+2}\right)\right)- \\
&-\left(1+\left(t_j+\varepsilon\right)\right)\left(P_3\left(t_{j-1}\right)-P_3\left(t_{j+2}\right)\right)+ \\
&+\frac{1}{2}\left(P_4\left(t_{j+2}\right)-P_4\left(t_{j-1}\right)+P_2\left(t_{j-1}\right)-P_2\left(t_{j+2}\right)+\pi\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&A_{j-1}\left(t_j+\varepsilon\right) = A_{j-1}^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{2\pi\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[t_j\left(t_j+\varepsilon\right)\left(P_2\left(t_{j-1}\right)-P_2\left(t_{j+2}\right)\right)- \\
&-\left(1+\left(t_j+\varepsilon\right)\right)\left(P_3\left(t_j\right)-P_3\left(t_{j-1}\right)\right)- \\
&-\frac{1}{2}\left(P_4\left(t_{j-1}\right)-P_4\left(t_j\right)+P_2\left(t_j\right)-P_2\left(t_{j-1}\right)\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&A_j\left(t_j+\varepsilon\right) = A_j^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{\pi h\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[-t_{j-1}\left(t_j+\varepsilon\right)\left(P_2\left(t_j\right)-P_2\left(t_{j-1}\right)\right)-\left(h+\varepsilon\right)\left(P_3\left(t_j\right)-P_3\left(t_{j-1}\right)\right)+ \\
&+\frac{1}{2}\left(P_4\left(t_{j-1}\right)-P_4\left(t_j\right)+P_2\left(t_j\right)-P_2\left(t_{j-1}\right)+\right. \\
&+t_{j+1}\left(t_j+\varepsilon\right)\left(P_2\left(t_{j+1}\right)-P_2\left(t_j\right)\right)+\left(\varepsilon-h\right)\left(P_3\left(t_{j+1}\right)-P_3\left(t_j\right)\right)- \\
&-\frac{1}{2}\left(P_4\left(t_j\right)-P_4\left(t_{j+1}\right)+P_2\left(t_{j+1}\right)-P_2\left(t_j\right)\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&A_{j+1}\left(t_j+\varepsilon\right) = A_{j+1}^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{\pi h\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[-t_j\left(t_j+\varepsilon\right)\left(P_2\left(t_{j+1}\right)-P_2\left(t_j\right)\right)-\varepsilon\left(P_3\left(t_{j+1}\right)-P_3\left(t_j\right)\right)+ \\
&+\frac{1}{2}\left(P_4\left(t_j\right)-P_4\left(t_{j+1}\right)+P_2\left(t_{j+1}\right)-P_2\left(t_j\right)+\right. \\
&+t_{j+2}\left(t_j+\varepsilon\right)\left(P_2\left(t_{j+2}\right)-P_2\left(t_{j+1}\right)\right)+\left(2h-\varepsilon\right)\left(P_3\left(t_{j+2}\right)-P_3\left(t_{j+1}\right)\right)- \\
&-\frac{1}{2}\left(P_4\left(t_{j+1}\right)-P_4\left(t_{j+2}\right)+P_2\left(t_{j+2}\right)-P_2\left(t_{j+1}\right)\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&A_{j+2}\left(t_j+\varepsilon\right) = A_{j+2}^{(1)}\left(t_j+\varepsilon\right) - \frac{1}{\pi h\sqrt{1-\left(t_j+\varepsilon\right)^2}} \times \\
&\times\Big[-t_{j+1}\left(t_j+\varepsilon\right)\left(P_2\left(t_{j+2}\right)-P_2\left(t_{j+1}\right)\right)+ \\
&+\left(h-\varepsilon\right)\left(P_3\left(t_{j+2}\right)-P_3\left(t_{j+1}\right)\right)+ \\
&+\frac{1}{2}\left(P_4\left(t_{j+1}\right)-P_4\left(t_{j+2}\right)+P_2\left(t_{j+2}\right)-P_2\left(t_{j+1}\right)\right)\Big].
\end{aligned}
$$

The coefficients $A_k^{(1)}\left(t_j+\varepsilon\right), k=0,\dots,N+1$, are found in (Eshkuvatov, Z.K. et al, 2007), which are:

$$
\begin{aligned}
&A_k^{(1)}\left(t_j+\varepsilon\right) = \frac{\sqrt{1-(t_j+\varepsilon)^2}}{\pi}\frac{h}{\sqrt{1-t_k^2}\left(t_k-\left(t_j+\varepsilon\right)\right)}, \\
&k=1,\dots j-2, j+3,\dots N,
\end{aligned}
$$

$$
\begin{aligned}
&A_0^{(1)}\left(t_j+\varepsilon\right) = \frac{1}{2\pi}\left[\left(1-(t_j+\varepsilon)\right)\ln\left|\frac{P_1(t_{j-1},t_j+\varepsilon)}{P_1(t_{j+2},t_j+\varepsilon)}\right| - \right. \\
&\left.-\sqrt{1-(t_j+\varepsilon)^2}\left(\pi+P_2(t_{j-1})-P_2(t_{j+2})\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&A_{N+1}^{(1)}\left(t_j+\varepsilon\right) = \frac{1}{2\pi}\left[\left(1+(t_j+\varepsilon)\right)\ln\left|\frac{P_1(t_{j-1},t_j+\varepsilon)}{P_1(t_{j+2},t_j+\varepsilon)}\right| + \right. \\
&\left.\sqrt{1-(t_j+\varepsilon)^2}\left(\pi+P_2(t_{j-1})-P_2(t_{j+2})\right)\right],
\end{aligned}
$$

$$A_{j-1}^{(1)}(t_j+\varepsilon) = -\frac{1}{\pi h}\left[\varepsilon \ln\left|\frac{P_1(t_j,t_j+\varepsilon)}{P_1(t_{j-1},t_j+\varepsilon)}\right| + \right.$$
$$\left. +\sqrt{1-(t_j+\varepsilon)^2}\left(P_2(t_j)-P_2(t_{j-1})\right)\right]$$

$$A_j^{(1)}(t_j+\varepsilon) = \frac{1}{\pi h}\left[(h+\varepsilon)\ln\left|\frac{P_1(t_j,t_j+\varepsilon)}{P_1(t_{j-1},t_j+\varepsilon)}\right| + \right.$$
$$+(h-\varepsilon)\ln\left|\frac{P_1(t_{j+1},t_j+\varepsilon)}{P_1(t_j,t_j+\varepsilon)}\right| +$$
$$\left. +\sqrt{1-(t_j+\varepsilon)^2}\left(2P_2(t_j)-P_2(t_{j-1})-P_2(t_{j+1})\right)\right],$$

$$A_{j+1}^{(1)}(t_j+\varepsilon) = \frac{1}{\pi h}\left[\varepsilon \ln\left|\frac{P_1(t_{j+1},t_j+\varepsilon)}{P_1(t_j,t_j+\varepsilon)}\right| + \right.$$
$$+(2h-\varepsilon)\ln\left|\frac{P_1(t_{j+2},t_j+\varepsilon)}{P_1(t_{j+1},t_j+\varepsilon)}\right| +$$
$$\left. +\sqrt{1-(t_j+\varepsilon)^2}\left(2P_2(t_{j+1})-P_2(t_j)-P_2(t_{j+2})\right)\right],$$

$$A_{j+2}^{(1)}(t_j+\varepsilon) = \frac{1}{\pi h}\left[(\varepsilon-h)\ln\left|\frac{P_1(t_{j+2},t_j+\varepsilon)}{P_1(t_{j+1},t_j+\varepsilon)}\right| + \right.$$
$$\left. +\sqrt{1-(t_j+\varepsilon)^2}\left(P_2(t_{j+2})-P_2(t_{j+1})\right)\right]$$

Estimate of errors

Theorem 1: Let $f(t)\in H^\alpha(A,[-1,1])$, and E be a set of canonic partition of the interval $[-1,1]$. Then the error of quadrature formula (12) is

$$|R_N(t_j+\varepsilon)| \le \begin{cases} L_1 h^\alpha \ln(N+1) + \dfrac{L_2}{\sqrt{1-(t_j+\varepsilon)^2}} h^\alpha, & \varepsilon = \dfrac{h}{2}, \\ L_1 h^\alpha \ln(N+1) + \dfrac{L_2}{\sqrt{1-(t_j+\varepsilon)^2}} h^\alpha + L_3 h^\delta, & \varepsilon\in(0,h), \varepsilon \ne \dfrac{h}{2}. \end{cases}$$

where

$$L_1 = \frac{6A}{\pi}\left(1+\frac{4.18}{\alpha\ln(N+1)}\right),\quad L_2 = 8A\left(1+h^{\frac{1}{2}}\right),$$
$$L_3 = A\left(0.567h^{2-\delta}+0.068h^{5-\delta}+0.516\right),$$
$$0<\delta\le \log_h \frac{|2h-\varepsilon|}{(2h-\varepsilon)(h+\varepsilon)}.$$

Theorem 2: Let $f(t)\in C^1([-1,1])$, and E be a set of canonic partition of the interval $[-1,1]$. Then the error of the quadrature formula (12) satisfies the inequality

$$|R_N(t_j+\varepsilon)| \le \begin{cases} L_1^* h\ln(N+1) + \dfrac{L_2^*}{\sqrt{1-(t_j+\varepsilon)^2}} h, & \varepsilon = \dfrac{h}{2}, \\ L_1^* h\ln(N+1) + \dfrac{L_2^*}{\sqrt{1-(t_j+\varepsilon)^2}} h^\alpha + L_3^* h^\delta, & \varepsilon\in(0,h), \varepsilon \ne \dfrac{h}{2}. \end{cases}$$

where

$$L_1^* = \frac{4M_1}{\pi}\left(1+\frac{2.15}{\alpha\ln(N+1)}\right),\quad L_2^* = 5M_1\left(1+h^{\frac{1}{2}}\right),$$
$$L_3^* = M_1\left(0.567h^{2-\delta}+0.068h^{5-\delta}+0.516\right),$$
$$0<\delta\le \log_h \frac{|2h-\varepsilon|h}{(2h-\varepsilon)(h+\varepsilon)},$$
$$M_1 = \max_{-1\le t\le 1}|f'(t)|.$$

The proof of the theorems 1 and 2 are based on the following lemmas.

Lemma 1. Let $f(t)$ be continuous function and the function $\varphi(t)$ be defined by (10). If $f(t)\in H^\alpha(A,[-1,1])$ then for any $t',t'',t\in[-1,1]$ the following estimates are true

$a)\ |\varphi(t'')-\varphi(t')| \le 2A|t''-t'|^\alpha;$

$b)\ |\varphi(t)| \le A(1-t^2)^\alpha.$

If $f(t)\in C^1([-1,1])$, then for any $t',t'',t\in[-1,1]$, the following estimations are hold:

$c)\ |\varphi(t'')-\varphi(t')| \le 2M_1|t''-t'|;$

$d)\ |\varphi(t)| \le M_1(1-t^2).$

Lemma 2 Let $f(t)$ be continuous function, and the function $s_\nu(t)$ be linear spline defined by (5). Then the following estimates

$a)$ If $f(t)\in H^\alpha(A,[t_\nu,t_{\nu+1}])$, then for any $t\in[t_\nu,t_{\nu+1}]$,

$$|f(t)-s_\nu(t)| \le 2^{-\alpha}Ah^\alpha$$

$b)$ If $f(t)\in C^1([t_\nu,t_{\nu+1}])$, then for any $t\in[t_\nu,t_{\nu+1}]$,

$$|f(t)-s_\nu(t)| \le \tilde{M}_1 h/2$$

are hold, where $\tilde{M}_1 = \max_{t_\nu\le\theta\le t_{\nu+1}}|f'(\theta)|$.

Lemmas 1 and 2 are proved in (Israilov M.I. and Eshkuvatov, Z.K. 1994 and Eshkuvatov, Z.K. and Nik Long, 2007 .).

Let

$$\psi_1(t)=(t+t_j+\varepsilon)\varphi(t), \quad (17)$$

$$\psi_2(t)=(t+t_j+\varepsilon)(f(t)-s_v(t)). \quad (18)$$

Lemma 3 For any $t\in[-1,1]$ the function $\psi_1(t)$ satisfying the following inequality

$a)$ If $\varphi(t)\in H^\alpha(2A,[-1,1])$ then

$$|\psi_1(t)|\le 2A(1-t^2)^\alpha.$$

$b)$ If $\varphi(t)\in C^1([-1,1])$ then

$$|\psi_1(t)|\le 2M_1(1-t^2).$$

Lemma 4 For any $t\in[t_k,t_{k+1}]$ the function $\psi_1(t)$ satisfies the following inequalities

$a)$ If $\varphi(t)\in H^\alpha(2A,[-1,1])$ then

$$|\psi_1(t)-\psi_1(t_k)|\le 2A(2+h)(1-t^2)^\alpha.$$

$b)$ If $\varphi(t)\in C^1([-1,1])$ then

$$|\psi_1(t)-\psi_1(t_k)|\le M(2+h)(1-t^2).$$

Lemma 5 For any $t\in[t_v,t_{v+1}]$ the function $\psi_1(t)$ satisfying the following inequality

$a)$ If $\varphi(t)\in H^\alpha(2A,[-1,1])$ then

$$|\psi_2(t)|\le 2^{1-\alpha}Ah^\alpha.$$

$b)$ If $\varphi(t)\in C^1([-1,1])$ then

$$|\psi_2(t)|\le \frac{1}{2}Mh.$$

The proof of the Lemmas 3 and 4 follows by Lemma 1, whence Lemma 5 is proved by the help of Lemma 2.

Proof of the Theorem 1. From (9), it follows that

$$|R_N(t_j+\varepsilon)|=\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\left|\int_{-1}^{1}\frac{\sqrt{1-t^2}\varphi(t)}{t-(t_j+\varepsilon)}dt-\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{\sqrt{1-t_k^2}\varphi(t_k)}{t_k-(t_j+\varepsilon)}h-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{\sqrt{1-t_k^2}s_v^*(t)}{t-(t_j+\varepsilon)}dt\right|$$

$$=\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\left|\left(1-(t_j+\varepsilon)^2\right)\left[\int_{-1}^{1}\frac{\varphi(t)}{\sqrt{1-t^2}(t-(t_j+\varepsilon))}dt-\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{\varphi(t_k)h}{\sqrt{1-t_k^2}(t_k-(t_j+\varepsilon))}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{s_v^*(t)}{\sqrt{1-t^2}(t-(t_j+\varepsilon))}dt\right]-\left[\int_{-1}^{1}\frac{(t+t_j+\varepsilon)\varphi(t)}{\sqrt{1-t^2}}dt-\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{(t_k+t_j+\varepsilon)\varphi(t_k)h}{\sqrt{1-t_k^2}}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{(t+t_j+\varepsilon)s_v^*(t)}{\sqrt{1-t^2}}dt\right]\right|\le R_1(t_j+\varepsilon)+R_2(t_j+\varepsilon),$$

Where

$$|R_1(t_j+\varepsilon)|=\frac{\sqrt{1-(t_j+\varepsilon)^2}}{\pi}\left|\int_{-1}^{1}\frac{\varphi(t)dt}{\sqrt{1-t^2}(t-(t_j+\varepsilon))}-\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{\varphi(t_k)h}{\sqrt{1-t_k^2}(t_k-(t_j+\varepsilon))}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{s_v^*(t)dt}{\sqrt{1-t^2}(t-(t_j+\varepsilon))}\right|,$$

$$|R_2(t_j+\varepsilon)|=\frac{1}{\pi\sqrt{1-(t_j+\varepsilon)^2}}\left|\int_{-1}^{1}\frac{(t+(t_j+\varepsilon))\varphi(t)dt}{\sqrt{1-t^2}}-\sum_{\substack{k=1,\\ k\notin Q(j)\cup j+2}}^{N}\frac{(t_k+(t_j+\varepsilon))\varphi(t_k)h}{\sqrt{1-t_k^2}}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{(t-(t_j+\varepsilon))s_v^*(t)dt}{\sqrt{1-t^2}}\right|$$

.

The estimate of $R_1(t_j+\varepsilon)$ has been proven in (Eshkuvatov, Z.K. et al, 2007), that is stated if $\varphi\in H^\alpha(2A,[-1,1])$, then $R_1(t_j+\varepsilon)$ is

$$|R_1(t_j+\varepsilon)|\le\begin{cases}L_1h^\alpha\ln(N+1), & \varepsilon=\frac{h}{2},\\ L_1h^\alpha\ln(N+1)+L_2h^\delta, & \varepsilon\in(0,h),\varepsilon\ne\frac{h}{2}.\end{cases}$$

where

$$L_1=\frac{24A}{\pi}\left(1+\frac{0.854}{\alpha\ln(N+1)}\right),\ L_2=\frac{A}{\pi}\left(0.68h^{5-\delta}+0.56h^{2-\delta}+0.516\right).$$

In oreder to estmate $R_2(t_j+\varepsilon)$ due to (17), (18), we write it as follows

$$\left|R_2\left(t_j+\varepsilon\right)\right| = \frac{1}{\pi\sqrt{1-\left(t_j+\varepsilon\right)^2}}\left|\int_{-1}^{1}\frac{\psi_1(t)}{\sqrt{1-t^2}}dt - \sum_{\substack{k=1,\\ k\notin Q(j)\cup j-2}}^{N}\frac{\psi_1(t_k)}{\sqrt{1-t_k^2}}h - \sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{\left(t+\left(t_j+\varepsilon\right)\right)s_v^*(t)}{\sqrt{1-t^2}}dt\right|.$$

In view of Lemmas 3*a*), 4*a*), 5*a*) the estimate of $R_2(t_j+\varepsilon)$ can be obtained in the form

$$\left|R_2\left(t_j+\varepsilon\right)\right| \le \frac{1}{\pi\sqrt{1-\left(t_j+\varepsilon\right)^2}}\left[\left|\int_{-1}^{t_1}\frac{\psi_1(t)}{\sqrt{1-t^2}}dt\right| + \left|\frac{\psi_1(t_{j+2})}{\sqrt{1-t_{j+2}^2}}h\right| + \left|\sum_{k=1,k\notin Q(j)}^{N}\int_{t_k}^{t_{k+1}}\frac{\psi_1(t)-\psi_1(t_k)}{\sqrt{1-t^2}}dt\right| + \left|\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{\psi_2(t)}{\sqrt{1-t^2}}dt\right| + \left|\sum_{k=1,k\notin Q(j)}^{N}\int_{t_k}^{t_{k+1}}\frac{\psi_1(t_k)\left(\sqrt{1-t_k^2}-\sqrt{1-t^2}\right)}{\sqrt{1-t^2}\sqrt{1-t_k^2}}dt\right|\right]$$

$$\le \frac{1}{\sqrt{1-\left(t_j+\varepsilon\right)^2}}\left[2\sqrt{2}Ah^{\left(\alpha+\frac{1}{2}\right)} + 2Ah + 4\pi Ah^{\alpha} + 3Ah + 4\pi Ah^{\alpha}\right]$$

$$\le 8\pi A\left[1+0.625h^{1-\alpha}+0.54h^{\frac{1}{2}}\right]h^{\alpha}$$

The prove of the Theorem 1 follows from R_1 and R_2.

Theorem 2 is proved in the same manner as Theorem 1, but we use second part of the Lemmas 1-5.

Numerical results

As the example, we consider the function $f(t)=t^2+t+1$. The exact solution is:

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,(t^2+t+1)dt}{t-x} = \frac{1}{\sqrt{1-x^2}}\left[-x^3-x^2-\frac{x}{2}+\frac{1}{2}\right].$$

ε = 0.9*h*		ε = *h*/4		ε = *h*/2		ε = 3*h*/7	
x	error	*x*	error	*x*	error	*x*	Error
-.882	0.09952	-.875	0.09594	-.870	0.08754	-.871	0.08994
-.862	0.07283	-.775	0.03965	-.850	0.06861	-.851	0.07113
-.842	0.05309	-.675	0.02338	-.830	0.05444	-.831	0.05713
-.042	0.04829	-.055	0.02751	-.050	0.00097	-.071	0.00659
-.002	0.04759	-.035	0.02795	-.030	0.00058	-.031	0.00740
0.278	0.03923	-.015	0.02837	-.010	0.00019	0.029	0.00858
0.358	0.03630	0.005	0.02877	0.010	0.00019	0.089	0.00967
0.778	0.04713	0.825	0.04456	0.830	0.05444	0.829	0.05172
0.818	0.06012	0.845	0.05935	0.850	0.06861	0.849	0.06606
0.858	0.08299	0.865	0.07874	0.870	0.08754	0.869	0.08511

Conclusion

Method of discrete vortices (MDV) invented and implimented by Belotserkovskii S.M, and Lifanov I.K. in 1956, has many application in different areas of Physics, Mechanics and aerodinamics. Its simplicity allows us to solve the problems easily. Therefore we develope MDV by using spline approximation and provided the convergence of the QF for any singular point $x=t_j+\varepsilon$ belonging to the segment $[t_j, t_{j+1}]$, j=1,...,N. we improve of the error of QF(16) in the classes of $H^{\alpha}(A,[-1,1])$ and $C^1([-1,t_1])$. The numerical results shows the validity of the QF(16) at different value of ε.

References

Belotserkovskii, S.M. and Lifanov, I.K. 1985. Numerical methods in singular integral equations. Moscow: Nauka, 254 p. (in Russian).

Chen, Y.Z. 2004. Singular Integral Equation Method for the Solution of Multiple Curved Cracks Problems, International Jounal of Solids and Structures. 41: 3505-3519.

Eshkuvatov, Z.K, Nik Long, N.M.A. and Abdulkawi, M. 2006. Approximating singular integral equations with bounded solution on the edges using quadrature formula . Proceedings of the National Conference on Mathematics and Statistics. UPM, 221-233.

Eshkuvatov, Z.K, Nik Long, N.M.A. 2007. Numerical evaluation for Cauchy type singular integral using modification of discrete vortex method and spline approximation. Malaysian journal of mathematical sciences.1 : 123-139.

Israilov, M.I. and Eshkuvatov, Z.K.1994. On a accurate of discrete vortex method for calculation of singular integrals. // Dokl. AN RUz. 6:3-7. (in Russian).

Kanwal, R.P.1997. Linear integral equations. Theory and technique. Academic press.

Lifanov, I.K., Poltavskii, L.N. and Vainikko, G.M. 2004. Hypersingular singular integral equations and their applications. CHAPMAN and HALL/CRC Press company.

Lifanov, I.K. and Polonskii, I.E. 1975. Proof of the numerical method of discrete vortices for solving singular integral equations. PMM. 1 39: 742-746.

Moiseiwitsch, B.L. 1977. Integral equation. Longman, USA.

Muskhelishvili, N.I. 1953. Singular Integral equations. Gostekhizda .1946. English translation- Noordhoff, Groningen, Holland.

Stechkin, S. and Subbotin, U. 1976. Spline in computational mathematics. Moskov: Nauka. 354.

Paramara orsi, A. 1990 .Spline approximation for Cauchy principal value integrals. Journal of computational and applied mathematics. 30:191-201.

Smith, H.V. 2006. The numerical evaluation of the error term in Gaussian quadrature rules. International Jounal of Mathematics education in science and technology. 37: 201-205.