J. King Saud Univ., Vol. 21, *Science* (Special Issue), pp. 41-48, Riyadh (2009/1430H.)

Exponential and Polynomial Decay in a Semilinear Integro-differential Elastic Equation

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Keywords and phrases: exponential decay, polynomial decay, relaxation function, semilinear, viscoelastic.
AMS Classification: 35B35, 35B40, 35L20.

Abstract. In this paper, we consider a semilinear integro-differential elastic equation, in a bounded domain, and show that the energy solution decays at the same rate of the decay of the relaxation function.

1 Introduction

In this paper, we consider the following semilinear problem

$$\begin{cases} u_{tt}(x,t)+Au(x,t)-\int_0^t g(t-\tau) \\ Au(x,\tau)d\tau+|u|^{\rho-2}u(x,t)=0, \\ \text{in } \Omega\times(0,\infty) \\ u(x,t)=0,\ x\in\partial\Omega,\ t\geq 0 \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \\ x\in\Omega, \end{cases} \tag{1.1}$$

where $A=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right)$, Ω is a bounded domain of $\mathbb{R}^n\ (n\geq 1)$ with a smooth boundary $\partial\Omega$, a_{ij} are bounded functions satisfying conditions to be specified later, g is a positive nonincreasing function defined on $\mathbb{R}^+$, and $\rho\geq 2$.

For $a_{ij}=\delta_{ij}$, Cavalcanti *et al.* [1] studied (1.1) in the presence of a localized damping cooperating with the dissipation induced by the viscoelastic term. Under the condition

$$-\xi_1 g(t)\leq g'(t)\leq -\xi_2 g(t),\ t\geq 0,$$

they obtained an exponential rate of decay. Berrimi *et al.* [2] improved Cavalcanti's result by showing that the viscoelastic dissipation alone is enough to stabilize the system. To prove their result, Berrimi *et al.* introduced a diffirent functional, which allowed them to weaken the conditions on g. This result has been later extended to a situation, where a source is competing with the viscoelastic dissipation, by Berrimi *et al.* [3]. Cavalcanti *et al.* [4], considered

$$u_{tt}-k_0\Delta u+\int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]d\tau+ b(x)h(u_t)+f(u)=0,$$

under similar conditions on the relaxation function g and $a(x)+b(x)\geq\delta>0$, and improved the result of [1]. They established an exponential stability when g is decaying exponentially and h is linear and a polynomial stability when g is decaying polynomially and h is nonlinear. Another problem, where the damping induced by the viscosity is acting on the domain and a part of the boundary, was also discussed by Cavalcanti et al. [5] and existence and uniform decay rate results were established. In the same direction, Cavalcanti *et al.* [6] have also studied, in a bounded domain, the following equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\,\Delta u(\tau)d\tau - \gamma\Delta u_t = 0, \tag{1.2}$$

for $\rho > 0$, and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This last result has been extended to a situation, where a source term is competing with the strong mechanism damping and the one induced by the viscosity, by Messaoudi and Tatar [7]. In their work, Messaoudi and Tatar combined the well depth method with the perturbation techniques to show that solutions with positive, but small, initial energy exist globally and decay to the rest state exponentially. Furthermore, Messaoudi and Tatar [8], [9] considered (1.2), for $\gamma = 0$, and established exponential and polynomial decay results in the absence, as well as in the presence, of a source term. We also mention the work of Kawashima and Shibata [10], in which a global existence and exponential stability of small solutions to a nonlinear viscoelastic problem has been established.

For nonexistence, Messaoudi [11] considered:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a u_t |u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega\times(0,\infty)$$

and showed, under suitable conditions on g, that solution with negative energy blow up in finite time if $\gamma > m$ and continue to exist if $m \geq \gamma$. This blow-up result has been pushed to certain situations, where the initial energy is positive, by Messaoudi [12]. A similar result was also proved, using a different method, by Wu [13].

In the present work, we generalize the earlier decay result to solutions of (1.1). The paper is organized as follows. In Section 2, we present some notations and material needed for our work and state a global existence theorem, which can be obtained following exactly the arguments of [6]. Section 3 contains the statement and the proof of our main result.

2 Preliminaries

In this section, we present some material needed for the proof of our result.

For the relaxation function g we assume

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = \ell > 0$$

(A2) There exists a positive constant ξ such that

$$g'(t) \leq -\xi g^p(t), \quad t \geq 0 \quad 1 \leq p < 3/2.$$

For the matrix $A = (a_{ij})$, we assume that

(A3) A is symmetric; i.e.

$$a_{ij} = a_{ji}, \quad \forall i, j = 1, 2, \cdots n, \quad \text{a.e. } x \text{ in } \Omega$$

(A4) A is positive definite; i.e. there exists a constant $\alpha_0 > 0$, for which

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \text{ in } \Omega$$

(A5) A is bounded; i.e.

$$|a_{ij}(x)| \leq M, \quad \forall i, j = 1, 2, \cdots n, \text{ a.e. } x \text{ in } \Omega.$$

(A6)
$$2 \leq \rho \leq \frac{2(n-1)}{(n-2)}, \quad n \geq 3$$
$$\rho \geq 2, \quad n = 1, 2$$

Proposition 2.1. *Assume (A1), (A3)- (A6) hold and let* $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ *be given . Then problem (1.1) has a unique global solution*

$$u \in C\big([0,\infty); H_o^1(\Omega)\big), \quad u_t \in C\big([0,\infty); L^2(\Omega)\big). \tag{2.1}$$

Remark 2.1. Conditions (A1), (A3), (A4) are necessary to guarantee the hyperbolicity of equation (1.1).

Remark 2.2. Condition (A6) is made so that the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$.

We introduce the "modified" energy functional

$$\varepsilon(t):=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)B(u(t))+\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{\rho}\|u\|_\rho^\rho, \tag{2.2}$$

Where

$$B(u(t))=\int_\Omega\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(t)}{\partial x_i}\frac{\partial u(t)}{\partial x_j}dx, \tag{2.3}$$

$$(g\circ\nabla u)(t)=\int_0^t g(t-s)B(u(t)-u(s))ds. \tag{2.4}$$

Lemma 2.2. Suppose that $v\in L^\infty(0,T;H^1(\Omega))$ and g *is a continuous function. Then we have, for* $0\le\theta\le 1$,

$$(g\circ v)(t)\le 2\left\{\left(\int_0^t g^{1-\theta}(s)ds\right)\|Bv(t)\|_{L^\infty(0,T)}\right\}^{\frac{p-1}{p-1+\theta}}\left((g^p\circ\nabla v)(t)\right)^{\frac{\theta}{p-1+\theta}} \tag{2.5}$$

And

$$(g\circ v)(t)\le\left\{\int_0^t Bv(s)ds+tBv(t)\right\}^{(p-1/p}\left((g^p\circ\nabla v)(t)\right)^{1/p}. \tag{2.6}$$

Proof. For $q\ge 1$ and $0\le\theta\le 1$, we have

$$(g\circ v)(t)=\int_0^t g^{\frac{1-\theta}{q}}(t-s)B^{\frac{1}{q}}(v(t)-v(s))g^{\frac{g-1+\theta}{q}}(t-s)B^{\frac{q-1}{q}}(v(t)-v(s))ds.$$

By applying Hölder's inequality, we get

$$(g\circ v)(t)\le\left(\int_0^t g^{1-\theta}(t-s)B(v(t)-v(s))ds\right)^{1/q}\left(g^{q-1+\theta}\circ\nabla v(t)\right)^{(q-1)/q}$$

By taking $q=(p-1+\theta)/p-1$, we obtain

$$(g\circ v)(t)\le\left(\int_0^t g^{1-\theta}(t-s)B(v(t)-v(s))ds\right)^{\frac{p-1}{p-1+\theta}}\left(g^p\circ\nabla v(t)\right)^{\frac{\theta}{p-1+\theta}}; \tag{2.7}$$

hence, estimate (2.5) follows easily for $0\le\theta\le 1$.

Finally, by taking $\theta=1$ in (2.7), estimate (2.6) is established.

3. Decay of solutions

In this section, we state and prove our main result. For this purpose we set:

$$F(t):=\varepsilon(t)+\varepsilon_1\psi(t)+\varepsilon_2\chi(t), \tag{3.1}$$

where ε_1 and ε_2 are positive constants and

$$\Psi(t):=\int_\Omega uu_t dx,\ \chi(t):=-\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\,dx \tag{3.2}$$

Lemma 3.1. *If* u *is a solution of (1.1) then the "modified" energy satisfies*

$$\varepsilon'(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)B(u(t))\le\frac{1}{2}(g'\circ\nabla u)(t)\le 0. \tag{3.3}$$

Proof. By multiplying equation in (1.1) by u_t and integrating over Ω, using integration by parts, hypotheses (A1)-(A5) and some manipulations as in [11], we obtain (3.3) for regular solutions. This inequality remains valid for weak solutions by a simple density argument.

Lemma 3.2. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \le \varepsilon(t) \le \alpha_2 F(t) \tag{3.4}$$

holds for two positive constants α_1 and α_2.

Proof. Similar manipulations as in [3], [8] give the desired result.

Lemma 3.3. *Under the assumptions* (A1)-(A6), the functional

$$\Psi(t) := \int_\Omega u u_t dx$$

satisfies, along solutions of (1.1),

$$\Psi'(t) \le \int_\Omega u_t^2 dx - \frac{\ell}{2} B(u(t)) + \frac{Mn}{2\alpha_0}\left(1 + \frac{Mn(1-\ell)}{\ell\alpha_0}\right)\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p \circ \nabla u\right)(t) - \|u\|_\rho^\rho \tag{3.5}$$

Proof.

By using equation in (1.1), we easily see that

$$\Psi'(t) = \int_\Omega u_t^2 dx - B(u(t)) + \sum_{i,j=1}^n \int_0^t \int_\Omega g(t-s)\alpha_{ij}(x) \frac{\partial u(t)}{\partial x_i}\frac{\partial u(s)}{\partial x_j} dx ds - \|u\|_\rho^\rho \tag{3.6}$$

We now estimate the third term in the RHS of (3.6) as follows:

$$\begin{aligned}
&\sum_{i,j=1}^n \int_\Omega \frac{\partial u(t)}{\partial x_i} \int_0^t g(t-s)a_{ij}(x)\frac{\partial u(s)}{\partial x_j} dx ds \\
&\le \frac{1}{2}B(u(t)) + \frac{1}{2}B\left(\int_0^t g(t-s)u(s)ds\right) \\
&\le \frac{1}{2}B(u(t)) \\
&+ \frac{1}{2}B\left(\int_0^t g(t-s)(u(s)-u(t)+u(t))ds\right) \\
&\le \frac{1}{2}B(u(t)) + \frac{1}{2}B\left(\int_0^t g(t-s)(u(s)-u(t))ds\right) \\
&+ \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_j}\right)ds\right) \\
&\times \left(\int_0^t g(t-s)\frac{\partial u(t)}{\partial x_j} ds\right) dx \\
&+ \frac{1}{2}B\left(\int_0^t g(t-s)u(t)ds\right)
\end{aligned} \tag{3.7}$$

We then use Young's inequality and (A5) to estimate the terms of (3.7). For the second term, we have:

$$\begin{aligned}
&\frac{1}{2}B\left(\int_0^t g(t-s)(u(s)-u(t))ds\right) \\
&\le \frac{M}{4}\left[\sum_{i,j=1}^n \int_\Omega \left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i}\right)ds\right)^2 dx + \sum_{i,j=1}^n \int_\Omega \left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_j} - \frac{\partial u(t)}{\partial x_j}\right)ds\right)^2 dx\right]
\end{aligned}$$

By using Cauchy-Schwarz inequality and (A4), we get:

$$\sum_{i,j=1}^{n}\int_\Omega\left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)ds\right)^2 dx$$
$$\leq n\left(\int_0^t g^{2-p}(s)ds\right)\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u|^2$$
$$dsdx \leq \frac{n}{\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right). \quad (3.8)$$

Therefore, we arrive at:

$$\frac{1}{2}B\left(\int_0^t g(t-s)(u(s)-u(t))ds\right)\leq \frac{nM}{2\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right)(t). \quad (3.9)$$

As for the third term, similar calculations and using the fact that $\int_0^t g(s)ds \leq 1-\ell$, gives, for $\eta>0$,

$$\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)ds\right)\times\left(\int_0^t g(t-s)\frac{\partial u(t)}{\partial x_j}ds\right)dx$$
$$\leq\frac{nM}{2\eta\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right)(t)+\frac{nM\eta(1-\ell)^2}{2\alpha_0}B(u(t)). \quad (3.10)$$

Finally, the fourth term can be handled as follows

$$\frac{1}{2}B\left(\int_0^t g(t-s)u(t)ds\right)=\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\left(\int_0^t g(t-s)\frac{\partial u(t)}{\partial x_i}ds\right)\left(\int_0^t g(t-s)\frac{\partial u(t)}{\partial x_j}ds\right)$$
$$dx=\frac{1}{2}\left(\int_0^t g(s)ds\right)^2\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\times\frac{\partial u(t)}{\partial x_i}\frac{\partial u(t)}{\partial x_j}dx\leq\frac{(1-\ell)^2}{2}B(u(t))). \quad (3.11)$$

By inserting (3.8)-(3.11) in (3.7), we get

$$\sum_{i,j=1}^{n}\int_0^t\int_\Omega g(t-s)a_{ij}(x)\frac{\partial u(t)}{\partial x_i}\frac{\partial u(s)}{\partial x_j}dxds$$
$$\leq\frac{Mn}{2\alpha_0}\left(1+\frac{1}{\eta}\right)\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right)(t) \quad (3.12)$$
$$+\left[\frac{1}{2}+\frac{n\eta M+\alpha_0}{2\alpha_0}(1-\ell)^2\right]B(u(t)).$$

By inserting (3.12) in (3.6) and taking $\eta=\alpha_0\ell/nM(1-\ell)$, (3.5) is established.

Lemma 3.4. *Under the assumptions (A1)-(A6), the functional*

$$\chi(t):=-\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along solutions of (1.1) and for any $\delta>0$,

$$\chi'(t)\leq\left\{\delta-\int_0^t g(s)ds\right\}\int_\Omega u_t^2dx+$$
$$\left(\frac{n\delta M}{\alpha_0}+\frac{\delta}{\alpha_0}C_*^{2\rho-2}\left(\frac{\varepsilon_0}{\ell}\right)^{(\rho-2)}\right)B(u(t))+ \quad (3.13)$$
$$\left(\frac{Mn}{\alpha_0}\left(1+\frac{1}{4\delta}\right)+\frac{1}{4\delta\alpha_0}\right)\left(\int_0^t g^{2-p}(s)ds\right)$$
$$\left(g^p\circ\nabla u\right)(t)-\frac{g(0)C_p}{4\delta\alpha_0}(g'\circ\nabla u)(t).$$

where C_p *is the Poincaré constant and* C_* *is the embedding constant.*

Proof. Direct computations, using (1.1), yield

$$\chi'(t)=-\int_0^t g(s)ds\int_\Omega u_t^2dx-\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-$$
$$u(\tau))d\tau dx+\left(1-\int_0^t g(s)ds\right)\int_\Omega\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u(t)}{\partial x_j}$$
$$\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx+ \quad (3.14)$$
$$B\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)+$$
$$\int_\Omega|u|^{\rho-2}u\int_0^t g(t-s)(u(t)-u(s))dsdx.$$

We proceed now to estimate the terms of (3.14). So, By using Cauchy-Schwarz and Poincaré's inequalities, the second term can be handled as follows:

$$-\int_\Omega u_t \int_0^t g'(t-\tau)\left(u(t)-u(\tau)\right)d\tau dx$$
$$\leq \delta\int_\Omega u^2{}_t dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g'(t-\tau)\left(u(t)-u(\tau)\right)d\tau\right)^2 dx \quad (3..15)$$
$$\leq \delta\int_\Omega u^2{}_t dx-\frac{g(0)C_p}{4\delta}\int_\Omega\int_0^t g'(t-\tau)\left|\nabla u(t)-\nabla u(\tau)\right|^2$$
$$d\tau dx\leq \delta\int_\Omega u^2{}_t dx-\frac{g(0)C_p}{4\delta\alpha_0}(g'\circ\nabla u)(t).$$

As for the third term, similar estimations give

$$\int_\Omega\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx$$
$$\leq Mn\delta\int_\Omega|\nabla u|^2 dx+\frac{Mn}{4\delta}\left(\int_0^t g^{2-p}(s)ds\right)\int_\Omega\int_0^t g^p(t-s) \quad (3.16)$$
$$|\nabla u(t)-\nabla u(s)|^2 dsdx\leq\frac{Mn\delta}{\alpha_0}B(u(t))+\frac{Mn}{4\delta\alpha_0}$$
$$\left(\int_0^t g^{2-p}(s)ds\right)(g^p\circ\nabla u)(t).$$

Also, by repeating similar calculations as in (3.7), (3.16), and using the fact that $\int_0^\infty g(s)ds\leq 1-\ell$, we estimate the fourth term of (3.14) as follows:

$$B\left(\int_0^t g(t-s)(u(s)-u(t))ds\right)\leq Mn\left(\int_0^t g^{2-p}(s)ds\right)$$
$$\times\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2 dsdx \quad (3.17)$$
$$\leq\frac{Mn}{\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)(g^p\circ\nabla u)(t)$$

Finally, using (A4), (A6), (2.2), and Lemma 3.1, the fifth term is estimated as follows

$$\int_\Omega |u|^{\rho-2}u\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$\leq\delta C_*^{2\rho-2}\|\nabla u\|_2^{2\rho-2}+\frac{1}{4\delta\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)$$
$$\left(g^p\circ\nabla u\right)(t)\leq\frac{\delta}{\alpha_0^{\rho-1}}C_*^{2\rho-2}\left[B(u(t))\right]^{\rho-1}+ \quad (3.18)$$
$$\frac{1}{4\delta\alpha_0}\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right)(t)$$
$$\leq\frac{\delta}{\alpha_0^{\rho-1}}C_*^{2\rho-2}\left(\frac{\varepsilon_0}{2\ell}\right)^{(\rho-2)}B(u(t))+\frac{1}{4\delta\alpha_0}$$
$$\left(\int_0^t g^{2-p}(s)ds\right)\left(g^p\circ\nabla u\right)(t).$$

By combining (3.14)-(3.18), the assertion of the lemma is proved.

Theorem 3.1. *Let $(u_0,u_1)\in H^1{}_0(\Omega)\times L^2(\Omega)$ be given Assume that (A1)-(A6) hold. Then, for each $t_0>0$, there exist strictly positive constants K and k such that the solution of (1.1) satisfies, for all $t\geq t_0$,*

$$\varepsilon(t)\leq Ke^{-kt},\quad p=1$$
$$\varepsilon(t)\leq K(1+t)^{-1/(p-1)},\quad p>1.$$

Proof

Since g is positive and $g(0)>0$ then for any $t_0>0$ we have

$$\int_0^t g(s)ds\geq\int_0^{t_0} g(s)ds=g_0>0,\quad\forall t\geq t_0. \quad (3.19)$$

By using (3.3), (3.5), (3.13), and (3.19), we obtain

$$F'(t)\leq-\left[\varepsilon_2(g_0-\delta)-\varepsilon_1\right]\int_\Omega u^2{}_t dx+$$
$$\left(\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta\alpha_0}C^2{}_2\right)(g'\circ\nabla u)(t)-$$
$$\left[\frac{\varepsilon_1\ell}{2}-\varepsilon_2\left(\frac{\delta nM}{\alpha_0}+\frac{\delta}{\alpha_0^{\rho-1}}C_*^{2\rho-2}\left(\frac{\varepsilon_0}{\ell}\right)^{(\rho-2)}\right)\right]B(u(t))+ \quad (3.20)$$
$$\left(\frac{\varepsilon_1 Mn}{2\alpha_0}\left(1+\frac{Mn(1-\ell)}{2\ell\alpha_0}\right)+\varepsilon_2\left(\frac{Mn}{\alpha_0}\left(1+\frac{1}{4\delta}\right)+\frac{1}{4\delta\alpha_0}\right)\right)\times$$
$$\left[\int_0^t g^{2-p}(\tau)d\tau\right]\left(g^p\circ\nabla u\right)(t)-\varepsilon_1\|u\|_\rho^\rho$$

At this point we choose δ so small that

$$g_0-\delta>\frac{1}{2}g_0,\quad \frac{2}{\ell}\left[\frac{\delta nM}{\alpha_0}+\frac{\delta}{\alpha_0^{\rho-1}}C_*^{2\rho-2}\left(\frac{\varepsilon_0}{\ell}\right)^{(\rho-2)}\right]<\frac{1}{4}g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2<\varepsilon_1<\frac{1}{2}g_0\varepsilon_2 \qquad (3.21)$$

will make

$$k_1=\varepsilon_2(g_0-\delta)-\varepsilon_1>0,\quad k_2=\frac{\varepsilon_1\ell}{2}-\varepsilon_2\left[\frac{\delta nM}{\alpha_0}+\frac{\delta}{\alpha_0^{\rho-1}}C_*^{2\rho-2}\left(\frac{\varepsilon_0}{\ell}\right)^{(\rho-2)}\right]>0.$$

We then pick ε_1 and ε_2 so small that (3.4) and (3.21) remain valid and, further,

$$-\frac{1}{\xi}\left(\frac{\varepsilon_1 Mn}{2\alpha_0}\left(1+\frac{Mn(1-\ell)}{2\ell\alpha_0}\right)+\varepsilon_2\left(\frac{Mn}{\alpha_0}\left(1+\frac{1}{4\delta}\right)+\frac{1}{4\delta\alpha_0}\right)\right)+\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta\alpha_0}C_p^2>0.$$

Therefore, we arrive at

$$F'(t)\le-\beta\left[\int_\Omega u_t^2dx+B(u(t))+(g^p\circ\nabla u)(t)+\|u\|_p^p\right],\forall t\ge t_0. \qquad (3.22)$$

Case 1. $p=1$:

We combine (3.4) and (3.22) to get

$$F'(t)\le-\beta_1\varepsilon(t)\le-\beta_1\alpha_1F(t),\ \forall t\ge t_0. \qquad (3.23)$$

A simple integration of (3.23) leads to

$$F(t)\le F(t_0)e^{\beta_1\alpha_1t_0}e^{-\beta_1\alpha_1t},\ \forall t\ge t_0. \qquad (3.24)$$

Thus (3.4), (3.24) yield

$$\varepsilon(t)\le\alpha_2F(t_0)e^{\beta_1\alpha_1t_0}e^{-\beta_1\alpha_1t}=Ke^{-kt},\ \forall t\ge t_0. \qquad (3.25)$$

Case 2. $p>1$:

By using (A1) and (A2) we easily see that

$$\int_0^\infty g^{1-\theta}(\tau)d\tau<\infty,\ \theta<2-p.$$

So Lemma 2.2 yields

$$(g\circ\nabla u)(t)\le C\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\varepsilon(0)\right\}^{(p-1)/(p-1+\theta)}\left\{(g^p\circ\nabla u)(t)\right\}^{\theta/(p-1+\theta)}.$$

Therefore we get, for $\sigma>1$,

$$\varepsilon^\sigma(t)\le C\varepsilon^{\sigma-1}(0)\left\{\int_\Omega u_t^2dx+B(u(t))+\|u\|_p^p\right\}+$$

$$\left\{(g\circ\nabla u)(t)\right\}^\sigma\le C\varepsilon^{\sigma-1}(0)\left\{\int_\Omega u_t^2dx+B(u(t))\right\}+ C\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\varepsilon(0)\right\}^{\sigma(p-1)/(p-1+\theta)}\times\left\{(g^p\circ\nabla u)(t)\right\}^{\sigma\theta/(p-1+\theta)}. \qquad (3.26)$$

By choosing

$\theta=\frac{1}{2}$ and $\sigma=2p-1$ (hence $\sigma\theta/(p-1+\theta)=1$), estimate (3.26) gives

$$\varepsilon^\sigma(t)\le C\left\{\int_\Omega u_t^2dx+B(u(t))+\|u\|_p^p+(g^p\circ\nabla u)(t)\right\}. \qquad (3.27)$$

By combining (3.4), (3.23) and (3.27), we obtain

$$F'(t)\le-\frac{\beta_2}{C}\varepsilon^\sigma(t)\le-\frac{\beta_2}{C}(\alpha_1)F^\sigma(t),\ \forall t\ge t_0, \qquad (3.28)$$

for some constant $\beta_2>0$. A simple integration of (3.28) over (t_0,t) leads to:

$$F(t)\le C(1+t)^{-1/(\sigma-1)},\ \forall t\ge t_0. \qquad (3.29)$$

As a consequence of (3.29), we have

$$\int_0^\infty F(t)dt+\sup_{t\ge0}tF(t)<\infty.$$

Therefore, by using Lemma 2.2 again, we have

$$g \circ \nabla u \le C\left[\int_0^t B(u(s))ds + tB(u(t))\right]^{(p-1)/p} \times \left(g^p \circ \nabla u\right)^{1/p} \le C\left[\int_0^t F(s)ds + tF(t)\right]^{(p-1)/p} \times \left(g^p \circ \nabla u\right)^{1/p} \le C(g^p \circ \nabla u)^{1/p},$$

which implies that

$$g^p \circ \nabla u \ge C\,(g \circ \nabla u)^p. \tag{3.30}$$

Consequently, a combination of (3.23) and (3.30) yields, $\forall t \ge t_0$,

$$F'(t) \le -C\left[\int_\Omega u_t^2(t)dx + B(u(t)) + (g \circ \nabla u)^p(t)\right]. \tag{3.31}$$

On the other hand, we have similar to (3.27),

$$\varepsilon^p(t) \le C\left[\int_\Omega u_t^2(t)dx + B(u(t)) + \|u\|_p^p + (g \circ \nabla u)^p(t)\right], \quad \forall t \ge t_0. \tag{3.32}$$

Combining the last two inequalities and (3.4), we obtain

$$F'(t) \le -CF^p(t), \; t \ge t_0. \tag{3.33}$$

A simple integration of (3.33) over (t_0, t) gives

$$F(t) \le K\,(1+t)^{-1/(p-1)}, \; \forall t \ge t_0. \tag{3.34}$$

This completes the proof.

Remark 3.1. Note that our result is proved without any condition on g unlike what was assumed in (2.4) of [6]. We only need g to be differentiable and satisfying (A1) and (A2).

Remark 3.2. Estimates (3.25) and (3.34) are also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $\varepsilon(t)$.

Acknowledgement. The authors would like to express their sincere thanks to King Fahd University of Petroleum and Minerals for its support. This work has been funded by KFUPM under Project # MS/DECAY/334.

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