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Stability and Hopf Bifurcations of Nonlinear Delay Malaria Epidemic Model

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Abstract. The objective of this paper is to systematically study the boundedness, persistence and stability of the nonlinear malaria epidemic model with latent periods. First, we consider the simplified model with the approximation $f(t-\eta)=f(t)-\eta f'(t)$, when η is small enough so that the function f does not vary too rapidly over the time interval [t-η, t],and study the stability of the trivial and the positive equilibrium points. Second, when the latent periods are equal (and not small enough), we will investigate the stability of the positive equilibrium point and prove the existence of Hopf Bifurcations and discuss the stability independent of the delays. Third, in the case when the latent periods are different, we will employ the Lyapunov functional method to establish some sufficient conditions for the local asymptotic stability of the positive equilibrium point.

Introduction

Mathematical models have been used to study the transmission and control of malaria since the first model that has been given by Ross in [23]. The Ross model consists of two nonlinear differential equations in two state variables that correspond to the proportions of infected human beings and the infected mosquitoes. The focus of the original work was malaria, but this work was extended to develop a rather general theory of disease transmission. George MacDonald [16] added a layer of biological realism to these early models by his careful attention to interpretation and estimation of the parameters. This work was based on Ross model and the collection, analysis and interpretation of epidemiological data malaria infection by the WHO project by Molieaux, L. and Gromicca, G. 1980. Garrette-Jones, C. 1964 created the vectorial capacity of any dynamic model of malaria to quantify how effectively the mosquito population transmits malaria.

The quantitative character of much of this work is in part a consequence of the emphasis placed by MacDonald, G. 1957, in his simple model of the dynamics of malaria. The value of mathematical studies to the design of malaria transmission, the control programs and the interpretation of observed epidemiological tends has been a topic of considerable controversy. We refer the reader to the papers by (Bruce-Chwatt and Glonville [3], MacDonald [15] and Martini [17] and for more details we refer the reader to the book by May and Anderson [18].

The basic model of malaria as given in [18] consists of two nonlinear differential equations describing changes in the proportion of infected humans h and mosquitoes m. The model is given by:

$$\begin{aligned} h'(t) &= \left(ab\left(\overline{N}/N\right)\right)m(t)\left[1-h(t)\right]-\mu h(t) \\ m'(t) &= ach(t)\left[1-m(t)\right]-\delta m(t), \end{aligned} \tag{1.1}$$

where:

(I) N is the size of human populations, N is size of female mosquito population (the ratio N/N defines the number of female mosquitoes per humans host),

(II) a is a mosquito's rate of biting people (the number of bites per unit time).

(III) b is the proportion of the people that become infected when bitten by an infectious mosquito.

(IV) μ is the per capita rate of humans recovery from infection.

(V) c is the proportion of mosquitoes that become infected after biting an infectious person.

(VI) δ is the per capita rate of mosquitoes mortality.

(VII) ($ab\overline{N}/N$)m(t) is the inoculation rate, the rate at which currently infectious mosquitoes deliver infecting bites.

The first equation of the system (1.1) describes the change in the proportion of infected humans and the second equation describes the change in the proportion of infected mosquitoes. In the first equation the term (1-h(t)) refers to the proportion of the population that are not infected, and the formulation implies that all of these noninfectious individuals are susceptible and can contract the infection. The essential feature of infection that is incorporated in these equations is that infection of a human occurs whenever an infective mosquito bits a healthy human, and infection of a mosquito occurs when a healthy mosquito bits an infective human.

In system (1.1) it is assumed that the total population of both humans and mosquitoes are constants, so that the dynamical variable are the proportion infected in each population h(t) and m(t). The model assumes that there is no vertical transmission of the parasite; i.e., all newborn humans and mosquitoes are susceptible. In addition, Ross concluded that the death rate in humans was negligible in comparison with their recovery rate and that the opposite held true for mosquitoes; i.e., the recovery rate in mosquitoes was negligible in comparison with their death rate. There are a number of assumptions that limit the utility of this formulation. Because there is also recovery from the infection (represented by the term μh), this further implies that there is no immunity to reinfection (May, R. M. and Anderson R. M. 1995). The model in its present form falls into the SIS class of epidemic models. Also the model assumes that an individual infected with malaria could not be again infected until after complete recovery from the initial infection. Let square brackets [.] indicate units. Then

$[\delta]=\text{time}^{-1}$, $[\mu]=\text{time}^{-1}$, $[\text{b}]=\text{people}\times\text{bites}^{-1}$,

$[\text{c}]=\text{mosquitoes}\times\text{bites}^{-1}$,

$[a]=time^{-1}\times mosquitoes^{-1/2}\times people^{1/2}\times[bc]^{-1/2}$,

$[\text{bc}] = \text{people}\times\text{mosquitoes}\times\text{bites}^{-2}$.

In this paper, we consider the modification of (1.1) to include incubation periods quoted from Ross, R. 1921 to be τ=0.5 month in human and σ=0.6 month in mosquito, i.e., we consider the nonlinear delay system

$$\begin{aligned} h'(t)&=\alpha m(t)[1-h(t)]-\gamma h(t)\\ m'(t)&=\beta h(t)[1-m(t)]-\delta m(t), \end{aligned} \tag{1.2}$$

Where:

$$\alpha=(ab\overline{N}/N),\quad \beta=ac,\quad \gamma=\mu+\mu,$$

where μ_1 is death rate in humans and αm(t-τ) is the new inoculation rate, the rate of uninfected humans that become infected, depends on the latent period of humans.

For epidemiological significance, we consider the initial conditions for (1.2) of the form:

$$h(\theta)=\varphi_1(\theta),\ m(\theta)=\varphi_2(\theta),\ \theta\in[-\tau_1,0], \tag{1.3}$$

where $\tau_1=\max\{\sigma,\tau\}$ and assume that $\varphi_i\in BC[-\tau_1,0]$, $\varphi_i>0$, i=1,2 (i.e.. are bounded, continuous and nonnegative functions on $[-\tau_1,0]$. Let | . | be any norm of R^2 and denote by $\|\varphi\|=\sup_{u\in[-\tau_1,0]}\varphi(u)$, where $\varphi=(\varphi_1,\varphi_2)^T$. In the following we denote by S_H the set of nonnegative continuous and bounded functions on $[-\tau_1,0]$ such that $\|\varphi\|\le H$ where H is a positive constant. Let X(t)= $(h_t,m_t)^T$ be the vector of R^2 whose components are defined by:

$$h_t=h(t+\theta),\ \text{and}\ m_t=m(t+\theta),\ \theta\in[-\tau_1,0]. \tag{1.4}$$

According to (1.4), $h_0=\varphi_1(0)>0$ and $m_0=\varphi_2(0)>0$. The system (1.2) can be rewritten as:

$$X'(t)=F(X_t), \tag{1.5}$$

with initial condition $X_0=\varphi(\theta)\in S_H$, where the vector $F(X_t)$ is given by:

$$F(X_t)=(F_1(X),F_2(X))^T \tag{1.6}$$

Where:

$$F_1(X_t) = \alpha m(t-\tau)\left[1-h(t-\tau)\right]-\gamma h(t),$$
$$F_2(X_t) = \beta h(t-\sigma)\left[1-m(t-\sigma)\right]-\delta m(t).$$

Denote by X_H for $t \geq 0$ the set of $BC[-\tau_1,0]$ such that $\|X_t\| \leq H$ for some $H>0$. Of course $X_H(0) \equiv S_H$. A solution of (1.5) for $t \geq 0$ satisfying the initial conditions $X(0)=\varphi(0)$, is denoted by:

$$X(t) = X(t, t_0, \varphi). \tag{1.7}$$

It is clear that the solution (1.7) of (1.2) which starts at (0, 0) in $R^2_{+0} := \{u \in R^2: u_i > 0, i=1,2\}$ will remain in R^2_{+0}. (e.g Beretta, Y. and Takeuchi, Y. 1994). Provided that the solutions (1.7) of (1.2) are bounded for any $t \geq 0$, then the solution $X(t, t_0, \varphi)$ can be uniquely continued to $[0,\infty)$ with its properties of continuous dependence on initial conditions φ and of positivity of the solution. We remark that in the last decades there are some different models of malaria has been introduced and some studies form different angles has been considered. For convenience, we refer the reader to the papers Aron, J. L. 1988, Bruce-Chwatt, L. J. and Glonville, V. J. 1973, Feng, Z., Yi, Y. and Zhu, H. 2004, Gravenor, M. B. et al. 2002, Gupta, S. and Hill, A. V. S. 1995, T. R. Jones, T. R. 1997, May, R. M. and Anderson R. M. 1995, Martini, C. 1921, McKenzie, F. E. and Bossert W. H. 1997, Kwiatkowski, D. and Nowack, M. 1991, Nasell, I. 1991, Nedelman, J. 1985 and the references cited therein.

Remark 1.1. We note that when there is no incubation periods admit to the model, i.e., when $\tau=\sigma=0$, the system (1.2) becomes:

$$\begin{aligned} h'(t) &= h(t)f(h,m) \\ &= h(t)\left[-\gamma+\left(\left(\alpha m(t)\right)/\left(h(t)\right)\right)-\alpha m(t)\right], \\ m'(t) &= mg(h,m) \\ &= m(t)\left[-\delta+\left(\left(\beta h(t)\right)/\left(m(t)\right)\right)-\beta h(t)\right]. \end{aligned} \tag{1.8}$$

In this case, we have $\frac{\partial f}{\partial h} = f_h(h,m)<0$ and $\frac{\partial g}{\partial h} = g_m(h,m)<0$ for $h>0$ and $m>0$. Then by Bendixon-Dulac Theorem there is no periodic orbit of (1.8) in the interior of the first quadrant of the phase plane. In Section 3, we will see that the delays will change this case and there exists a Hopf bifurcation.

The paper is organized as follows: In Section 2, we examine the boundedness and persistence of the solution of the system (1.2). Also we consider the simplified model after substituting the approximations

$$h(t-\sigma) \simeq h(t)-\sigma h'(t),$$
$$h(t-\tau) \simeq h(t)-\tau h'(t),$$
$$m(t-\sigma) \simeq m(t)-\sigma m'(t),$$
$$m(t-\tau) \simeq m(t)-\tau m'(t),$$

and study the asymptotic stability of the equilibrium points. In Section 3, by employing the stability switch theorem due to Cooke, K. L. and Driessche, P. van den. 1986, we investigate the local asymptotic stability and prove that there exists a Hopf Bifurcation at the positive equilibrium point E_* of (1.2) when the delay increases and consequently there is a periodic oscillations induced by the delay. This shows that there exists a major effect of the incubation periods in the behavior of the model. Also, we discuss the stability independent of the delays when the delays are equal as well as when the delays are different. In Section 4, we establish some sufficient conditions for local asymptotic stability by employing the Lyupanov functional method Kuang, Y. 1993 when the incubation periods are different. The global asymptotic stability of the positive equilibrium point still open and this will be of our interest in future.

Persistence and a Simplified Model

In this Section, we examine the boundedness and persistence of the solution of the system (1.2). For more details about the persistence of biological and ecological systems, we refer the reader to the book Kuang, Y. 1993. Also, we consider the simplified model after substituting the approximations

$$h(t-\sigma) \simeq h(t)-\sigma h'(t),$$
$$h(t-\tau) \simeq h(t)-\tau h'(t),$$
$$m(t-\sigma) \simeq m(t)-\sigma m'(t),$$
$$m(t-\tau) \simeq m(t)-\tau m'(t),$$

and study the asymptotic stability of the equilibrium points.

Definition 2.1. *The system (1.2) is said to be*

persistent if every positive solution (h(t), m(t)) of (1.2) satisfies $\lim_{t\to\infty}\inf\ h(t)>0$ *and* $\lim_{t\to\infty}\inf\ m(t)>0$ *and the system (1.2) is said to be uniformly persistent if there exists two positive numbers* h_1 *and* m_1 *such that every positive solution (h(t), m(t)) satisfies*
$\lim_{t\to\infty}\inf\ h(t)>h_1>0$
and $\lim_{t\to\infty}\inf\ m(t)>m_1>0$

Definition 2.1. *We say that the solution (h(t), m(t)) of (1.2) is permanent if there exist positive constants* C_1, C_2, D_1 *and* D_2 *with* $0<C_1\leq C_2<\infty$ *and* $D_1\leq D_2<\infty$ *such that for any positive initial positive conditions there exists a positive integer* $T>0$ *which depends on the initial conditions such that* $C_1\leq h(t)\leq C_2$ *and* $D_1\leq m(t)\leq D_2$ *for* $t\geq T$.

From applications point of view permanence guarantees the long term of the diseases.

Theorem 2.1. *Let (h(t), m(t)) denote any positive solution of system (1.2).*

(1). *If:*

$$\limsup_{t\to\infty} m(t)<\infty, \qquad (2.1)$$

then

$$\limsup_{t\to\infty} h(t)<\infty, \qquad (2.2)$$

(2). *If* $\limsup_{t\to\infty} h(t)<\infty$, *then*

$$\limsup_{t\to\infty} m(t)<\infty, \qquad (2.3)$$

(3). *If there exist* $m_1>0$ *and* $M_1>0$ *such that*

$$m_1\leq \liminf_{t\to\infty} m(t)\leq \limsup_{t\to\infty} m(t)\leq M_1$$

then there exist positive constants h_1 *and* H_1 *(independent of the solutions) such that*

$$h_1\leq \liminf_{t\to\infty} h(t)\leq \limsup_{t\to\infty} h(t)\leq H_1. \qquad (2.4)$$

(4). *If there exist* $h_1>0$ *and* $H_2>0$ *such that*

$$h_1\leq \liminf_{t\to\infty} h(t)\leq \limsup_{t\to\infty} h(t)\leq H_2.,$$

then there exist positive constants m_1 *and* M_2 *(independent of the solutions) such that:*

$$m_1\leq \liminf_{t\to\infty} m(t)\leq \limsup_{t\to\infty} m(t)\leq M_2 \qquad (2.5)$$

Proof. We only prove (1) and (3), since the proofs of (2) and (4) are similar and hence the details are omitted. Let (h(t), m(t)) denote any positive solution of system (1.2). From (1.2), we see that $h'(t)\geq -\gamma h(t)$, which implies that:

$$\liminf_{t\to\infty} h(t)\geq \varphi_1(0)=m_1>0.$$

Also, we can see that

$$\liminf_{t\to\infty} m(t)\geq \varphi_2(0)=m_2>0.$$

Suppose that (2.1) holds, then we can see that there exist $M_1>0$ and $t_1>0$ such that:

$$0<m(t)\leq M_1,\ \text{for } t\geq t_1 \qquad (2.6)$$

From the first equation of (1.2), we see that $h'(t)\geq -\gamma h(t)$ and this implies for $t\geq t_1+2\max\{\tau,\sigma\}$ that:

$$h(t)\geq e^{-\gamma\tau}h(t-\tau),\ \text{and}\ h(t)\geq e^{-\gamma\sigma}h(t-\sigma). \quad (2.7)$$

Also, from the second equation of (1.2), we can see for $t\geq t_1+2\max\{\tau,\sigma\}$, that:

$$m(t)\geq e^{-\gamma\tau}m(t-\tau),\ \text{and}\ m(t)\geq e^{-\delta\sigma}m(t-\sigma). \quad (2.8)$$

Now, from (1.2), (2.6) and (2.8), we have:

$$h'(t)\leq \alpha e^{\delta\tau}m(t)[1-h(t-\tau)]-\gamma h(t)$$
$$\leq \alpha e^{\delta\tau}M_1-\gamma h(t),$$

for $t\geq t_1+2\max\{\tau,\sigma\}$. Thus:

$$h(t)\leq \frac{\alpha e^{\delta\tau}M_1}{\gamma}=:H_1 \qquad (2.9)$$

for $t\geq t_2=t_1+\max\{\tau,\sigma\}$. On the other hand, by the positivity invariance of the solutions, from (1.2) and (2.7), we have:

$$h'(t)\geq \alpha m_2[1-e^{\gamma\tau}h(t)]-\gamma h(t)$$
$$=\alpha m_2-[\alpha m_2 e^{\gamma\tau}+\gamma]h(t),$$

which implies that:

$$h(t)\geq \frac{\alpha m_2}{[\alpha m_2 e^{\gamma\tau}+\gamma]}=:h_1 \qquad (2.10)$$

for $t\geq t_3>t_2$. For the proof of case (3), we can see from the above derivation that if m_2 and M_2 are uniform lower and upper bounds of the second variable of all the positive solutions of system (1.2), then (2.9) and (2.10) also hold uniformly for all positive solution of h(t) of (1.2). The proof is complete.

Next, we consider the simplified model and study the stability of the equilibrium points. Most studies on biological and epidemiological systems start from the local stability analysis of some special solutions (equilibrium points). For this purpose, the standard approach is to analyze the stability of the linearized equations about the special solutions. If the delay differential equations are autonomous and the special solution is constant, then the linearized equation take the form of linear autonomous delay differential equation. The stability of the trivial solution (zero solution) of the linearized equation depends on the location of the roots of the corresponding characteristic equation. If the roots of the characteristic equation for the linearization at the equilibrium have negative real parts, and if all the roots are uniformly bounded away from the imaginary axis, then the trivial solution of the linear equation is uniformly asymptotically stable.

Now, we assume that the delays σ and τ are equal and small enough, this can be true if the disease persist for a long time, so that h(t) and m(t) do not vary two rapidly over the time interval [t-σ, t], one may approximate

$$h(t-\sigma)\simeq h(t-\tau)\simeq h(t)-\sigma h'(t),$$

and

$$m(t-\sigma)\simeq m(t-\tau)\simeq m(t)-\sigma m'(t).$$

With these approximations, and after appropriate algebraic rearrangement and considering the case $\alpha=\beta$ and $\gamma=\delta$, model (1.2) becomes:

$$\begin{aligned}&(1-\alpha\sigma m)h'(t)+\alpha\sigma(1-h)m'(t)\\&=\alpha m(1-h)-\gamma h,\\&\alpha\sigma(1-m)h'(t)+(1-\alpha\sigma h)m'(t)\\&=\alpha h(1-m)-\gamma m.\end{aligned}$$

Solving for h'(t) and m'(t), we have:

$$\begin{aligned}h'(t)&=f(h(t),m(t))\\&=\left(\frac{-Ah^2+Bhm+Ch-Dm}{(L(h+m)-M)}\right),\\m'(t)&=g(h(t),m(t))\\&=\left(\frac{Am^2-Bhm-Cm+Dh}{(L(h+m)-M)}\right),\end{aligned}\tag{2.11}$$

where

$$A=(\sigma\alpha^2+\sigma\gamma\alpha),\ B=(\alpha+\alpha\sigma\gamma),$$
$$C=(\sigma\alpha^2+\gamma),\ D=(\alpha+\alpha\sigma\gamma),$$
$$L=(\alpha\sigma-\alpha^2\sigma^2),\ \text{and } M=(1-\alpha^2\sigma^2).$$

A simple algebra shows that the system (2.11) has trivial steady state $(h_0,m_0)=(0,0)$, which is the free disease case, and the non-trivial steady state:

$$\begin{aligned}(h_1,\ m_1)&=\left(\frac{C-D}{A-B},\frac{C-D}{A-B}\right)\\&=\left(\frac{(\alpha-\gamma)}{\alpha},\frac{(\alpha-\gamma)}{\alpha}\right),\end{aligned}\tag{2.12}$$

provided that $\alpha>\gamma$. First, we consider the free disease case $(h_0, m_0)=(0,0)$. To analyze the stability of (h_0, m_0) the eigenvalues of its Jacobian matrix have to be investigated. The Jacobian matrix of system (2.11) at (h_0, m_0) is given by:

$$J_0=\begin{pmatrix}\frac{\partial f}{\partial h}(h_0,m_0) & \frac{\partial f}{\partial m}(h_0,m_0)\\ \frac{\partial g}{\partial h}(h_0,m_0) & \frac{\partial f}{\partial m}(h_0,m_0)\end{pmatrix}$$
$$=\begin{pmatrix}\frac{-(\sigma\alpha^2+\gamma)}{(1-\alpha^2\sigma^2)} & \frac{(\alpha+\alpha\sigma\gamma)}{(1-\alpha^2\sigma^2)}\\ \frac{-(\alpha+\alpha\sigma\gamma)}{(1-\alpha^2\sigma^2)} & \frac{(\sigma\alpha^2+\gamma)}{(1-\alpha^2\sigma^2)}\end{pmatrix}.$$

Thus the characteristic equation is given by:

$$\lambda^2+\left(\frac{(\alpha+\alpha\sigma\gamma)^2}{(\alpha^2\sigma^2-1)^2}-\frac{(\sigma\alpha^2+\gamma)^2}{(\alpha^2\sigma^2-1)^2}\right)=0,$$

and the the associated eigenvalues are:

$$\lambda_1=-\left(\frac{(\sqrt{(\alpha+\gamma)}\sqrt{(\alpha-\gamma)})}{(\sqrt{(\alpha\sigma-1)}\sqrt{(\alpha\sigma+1)})}\right),$$
$$\lambda_2=\left(\frac{(\sqrt{(\alpha+\gamma)}\sqrt{(\alpha-\gamma)})}{(\sqrt{(\alpha\sigma-1)}\sqrt{(\alpha\sigma+1)})}\right).$$

Now, we note that if α>γ and ασ>1, then the equilibrium point (h_0,m_0) is a saddle point and then unstable. But if α>γ and ασ<1 then the point (h_0,m_0) is a centre and then becomes stable.
From the above discussion, we have the following stability result of the trivial equilibrium point of the system (2.11).

Theorem 2.2. *Assume that α=β, γ=δ, and τ=σ is small enough. Then the point (h_0,m_0) is:*
(a). Unstable if α>γ and ασ>1,
(b). Stable if α>γ and ασ<1,

Next, we consider the endemic case (h_1, m_1). From (2.12) it is evident that for this non-trivial fixed point to be biologically meaningful the conditions α>γ and such that $(1-\alpha\sigma)>0$ must hold. Notice that the necessary condition $(1-\alpha\sigma)>0$ provides an upper bound for values of σ that are biologically meaningful, one has ασ<1, which implies that σ<(1/α).

For simplicity, we consider the case α=2γ. In this case the equilibrium point becomes $(h_1, m_1)=((1/2),(1/2))$ and the coefficients become:

$$A = 4\sigma\gamma^2,\ B=2\gamma(\sigma\gamma+1),\ C=\gamma(2\sigma\gamma+1),$$
$$D = 2\gamma(\sigma\gamma+1),\ L=2\sigma\gamma(1-\sigma\gamma),$$
$$M=(1-2\gamma^2\sigma^2).$$

The Jacobian matrix of system (2.11) at (h_1,m_1) is given by:

$$J_1 = \begin{pmatrix} \frac{\partial f(h_1,m_1)}{\partial h} & \frac{\partial f(h_1,m_1)}{\partial m} \\ \frac{\partial g(h_1,m_1)}{\partial h} & \frac{\partial f(h_1,m_1)}{\partial m} \end{pmatrix}.$$

Thus the characteristic equation is given by:

$$\lambda^2+\gamma^2(-4\sigma^5\gamma^5+6\sigma^4\gamma^4+2\sigma^3\gamma^3$$
$$-26\sigma^2\gamma^2+22\sigma\gamma-5)=0,$$

and the the associated eigenvalues are:

$$\lambda_1 = -\gamma\sqrt{(2x-1)}\sqrt{(2x^4-2x^3-2x^2+12x-5)},$$
$$\lambda_2 = \gamma\sqrt{(2x-1)}\sqrt{(2x^4-2x^3-2x^2+12x-5)},$$

Where $x=\sigma\gamma$. We note that when $\sigma=(1/(2\gamma))$, this implies that the eigenvalues are zeros. Noting that the function $f(x)=2x^4-2x^3-2x^2+12x-5$ is a positive function for $x>0.46$ and negative in the interval $0<x\le 0.46<1/2$, so that the value of the function f(x) is negative when $0<\gamma\sigma\le 0.46<1/2$. This implies that the eigenvalues are real numbers and

$$\lambda_2<0<\lambda_1.$$

So that the equilibrium point (h_1,m_1) is a saddle point and thus it is unstable. This means that in the case when α=2γ the equilibrium point is unstable. This biologically is very interesting and explain that, when the rate at which the currently infectious mosquitoes that deliver infecting bites is greater than the sum of recovery and death humans rates then there exists an endemic situation which is unstable.

From the above discussion, we have the following stability result of the positive equilibrium point of the system (2.11).

Theorem 2.3. *Assume that α=β, γ=δ, with α=2γ and τ=σ is small enough such that γσ<(1/2). Then the positive equilibrium point is a saddle point and thus unstable.*

Stability and Hopf Bifurcations

Time delay plays an important role in many biological and epidemiological dynamical systems. When the delays are finite, the characteristic equations are functions of delays. As lengths of delays changes, the stability of the trivial solution may also changes. Such phenomena is often refereed to as stability switches. In this Section, we discuss the local asymptotic stability of the unique positive equilibrium point (h^*,m^*), the existence of Hopf Bifurcations and discuss the stability independent of the delays when the delays are not small. In order to analyze the full characteristic equation, we make use of the following Lemma which is the modification of the result by Cooke, K. L. and Driessche, P. van den. 1986. The Lemma is extracted from the book by Kuang, Y. 1993

Lemma 3.1 [Kuang, Y. 1993]. *Consider the characteristic equation of the form*

$P(\lambda)+Q(\lambda)e^{-\lambda\tau}=0$, and define $F(\omega)=|P(i\omega)|^2-|Q(i\omega)|^2$. Suppose $P(\lambda)$ and $Q(\lambda)$ have no common imaginary zeros, $P(0)+Q(0)\neq 0$, $P(-i\omega)=P(iw)$, $Q(-i\omega)=Q(i\omega)$ for real w and $F(\omega)$ has at most a finite number of real zeros. Then if $F(w)$ has no positive real root then there are no stability switches as τ increases, while stability switches are possible if $F(\omega)$ has at least one positive roots.

The system (1.2) admits both the trivial equilibrium $E_0=(h_0, m_0)=(0, 0)$ which corresponds to the free disease case when $(\alpha/\delta)=(\gamma/\beta)$, and the positive equilibrium point:

$$E_+=(h^*,m^*)=\left(\frac{\alpha\beta-\gamma\delta}{(\alpha+\gamma)\beta},\frac{\alpha\beta-\gamma\delta}{\alpha(\beta+\delta)}\right)$$

provided that $(\alpha/\delta)>(\gamma/\beta)$, which corresponds to the endemic case. Let:

$$h(t)=x(t)+h^* \text{ and } m(t)=y(t)+m^*, \qquad (3.1)$$

where (h^*,m^*) is the unique positive equilibrium point of (1.2). Substituting from (3.1) into (1.2), we find that $X(t)=(x(t),y(t))^T$ satisfies:

$$\begin{aligned}&\frac{dx(t)}{dt}=\alpha[y(t-\tau)+m^*][1-x(t-\tau)-h^*]\\&-\gamma x(t)-\gamma h^*,\\&\frac{dy(t)}{dt}=\beta[x(t-\sigma)+h^*][1-y(t-\sigma)-m^*]\\&-\delta y(t)-\delta m^*.\end{aligned} \qquad (3.2)$$

Then, the variational system of (1.2) with respect to the positive equilibrium point (h^*,m^*) is given by the linearized system of (3.2), that is:

$$\begin{aligned}&\frac{dx(t)}{dt}+\gamma x(t)+\alpha m^* x(t-\tau)\\&+\alpha(h^*-1)y(t-\tau)=0,\\&\frac{dy(t)}{dt}+\delta y(t)+\beta h^* y(t-\sigma)\\&+\beta(m^*-1)x(t-\sigma)=0.\end{aligned} \qquad (3.3)$$

The characteristic equation corresponding to the system (3.3), is given by:

$$\begin{aligned}&\lambda^2+\lambda A+C_1\lambda e^{-\lambda\tau}+C_2\lambda e^{-\lambda\sigma}+D_1e^{-\lambda\tau}\\&+D_2e^{-\lambda\sigma}+Ee^{-\lambda(\sigma+\tau)}+B=0\end{aligned}, \qquad (3.4)$$

where
$A=\gamma+\delta$, $B=\delta\delta$, $C_1=\alpha m^*$, $C_2=\beta h^*$,
$D_1=\alpha\delta m^*$, $D_2=\beta\gamma h^*$, $E=\alpha\beta[m^*+h^*-1]$.

We consider the effect of the delays and analyze the characteristic equation (3.4) in two different cases:

Case (1): $m^*+h^*=1$ and $\tau=\sigma=\tau$,

Case (2): $m^*+h^*-1=h^*m^*$ and $\tau\neq\sigma$.

First, we consider Case (1). In this case, the characteristic equation (3.4) becomes:

$$\lambda^2+\lambda A+B+e^{-\lambda\tau}\,[C\lambda+D]=0, \qquad (3.5)$$

where $C=C_1+C_2$ and $D=D_1+D_2$.

Remark. 3.1. We note that in the case when $\tau=0$, the roots of (3.5) are real and negative since $A+C>0$, so that the fixed point (h^*,m^*) is asymptotically stable.

In order to understand the stability switches of (3.5) in detail, it is crucial to determine the value of τ^* at which (3.5) may have a pair of conjugate imaginary roots, where in the work of Cooke, K. L. and Driessche, P. van den. 1986, τ is regarded as variable which may increase from zero to ∞. If the roots of (3.5) are in the left-half plane for $0\le\tau<\infty$, implying that the equilibrium is asymptotically stable for all τ, the equilibrium is said to be absolutely stable. Otherwise, there may be values of τ for which pairs of complex conjugate roots of the (3.5) cross the imaginary axis. If the crossing is from left to right, thus the equilibrium is destabilized, and if the crossing from right to left an unstable equilibrium may be stabilized as τ increases.

We assume that $\lambda=i\omega$, $\omega>0$ is a root of (3.5) for some $\tau>0$. In (3.5), we denote:

$$P(\lambda):=\lambda^2+\lambda A+B, \text{ and } Q(\lambda):=C\lambda+D.$$

It is clear that $P(0)+Q(0)\neq 0$, this means that λ=0 is not a root of (3.5). This implies that:

$P(i\omega)=(B-\omega^2)+iA\omega$, and $Q(i\omega)=D+iC\omega$.

Hence

$$F(\omega):=|P(i\omega)|^2-|Q(i\omega)|^2$$
$$=\omega^4-(C^2+2B-A^2)\omega^2+(B^2-D^2).$$

It is clear that, there is a positive real root w of F(ω)=0 if and only if there is a positive real root u= ω^2 of the equation:

$$u^2-(C^2-A^2+2B)u+(B^2-D^2)=0. \qquad (3.6)$$

From (3.6) it is clear that there exists a positive real root if B<D, while if $(C^2+2B-A^2)^2<4(B^2-D^2)$ the roots of (3.6) are non-real and then no stability switches may occur.

If B>D and

$$(C^2+2B-A^2)^2>4(B^2-D^2)$$

the roots of (3.6) are real, and are both positive if $(C^2+2B-A^2)<0$, both negative if $(C^2+2B-A^2)>0$.

Thus there is a positive real root of (3.6) if and only if either:

(h_1). B<D, or

$B>D$, $(C^2+2B-A^2)^2>4(B^2-D^2)$,

and $(C^2+2B-A^2)<0$,

which is equivalent to:

(h_2). $(C^2+2B-A^2)<-2\sqrt{(B^2-D^2)}$.

So according to Lemma 3.1 the stability switches are possible and it is possible for roots of the characteristic (3.5) to cross the imaginary axis, and this crossing are from the left to the right if F'(ω)>0, and from right to the left if $F'(\omega)<0$ at a crossing $i\omega$. Then, if $B<D$, $F(\omega)=0$ has a single positive root and it is destabilizing. If B>D there are two positive roots ω_+ and ω_- of F(ω)=0 given by

$$\omega^2_+=\frac{1}{2}\{(C^2+2B-A^2)^2+\Lambda^{(1/2)}\},$$
$$\omega^2_-=\frac{1}{2}\{(C^2+2B-A^2)^2-\Lambda^{(1/2)}\},$$

where

$\Lambda:=(C^2+2B-A^2)^2-4(B^2-D^2)$, with $\omega_-<\omega_+$. Therefore the following holds

$$2\omega^2_\pm-(C^2-A^2+2B)=\pm\Lambda^{(1/2)}.$$

This implies that

$$sign(F'(\omega))=2\omega_\pm sign\{\pm\Lambda^{(1/2)}\},$$

where $\omega_\pm>0$. It is clear that the sign is positive for ω_+ and negative for ω_-. From (3.5), when λ=i ω, we see that w satisfies the equations

$$C\omega\sin(\omega\tau)+D\cos(\omega\tau)+(B-\omega^2)=0,$$
$$-D\sin(\omega\tau)+C\omega\cos(\omega\tau)+A\omega=0,$$

which after simplification imply that

$\sin(\omega_\pm\tau):=((\omega_\pm{}^2-B)\omega_\pm C-AD\omega_\pm)/(D^2+\omega_\pm{}^2C^2))$,

$\cos(\omega_\pm\tau):=(((D-AC)\omega_\pm{}^2-BD)/(D^2+\omega_\pm{}^2C^2))$.

Hence there are $\theta_\pm$, $0<\theta_\pm\leq 2\pi$ such that $\theta_\pm=\omega_\pm\tau$, and

$$\theta_\pm:=\arctan\frac{((\omega_\pm{}^2-B)\pm C-AD\pm)}{((D-AC)\omega_\pm{}^2-BD)}.$$

In the case of B<D, only one imaginary root exists, λ=iw, therefore, the only crossing of the imaginary axes is from left to right as τ increases, and the stability of the trivial solution can only be lost and not regained.

In the case B>D, crossing from left to right with increasing τ occurs whenever τ assumes a value corresponding to w_+, and crossing from right to left occurs for values of τ corresponding to w_-. Then according to w_+ and w_-we obtain the following two sets of values of τ for which there are imaginary roots:

$$\tau^{\pm}{}_j := \frac{\theta_{\pm} + 2\pi j}{\omega_{\pm}}, \quad \text{for j=0, 1,...} .$$

In the case of (h_1) only $\tau^+{}_0$ needed be considered, since the zero solution is asymptotically stable for τ=0, where A+C>0. Then the zero solution remains asymptotically stable until τ_0^+, and unstable thereafter. At the value of $\tau^+{}_0$ (3.5) has pure imaginary roots $i\,\omega_{\pm}$.

In the case (h_2), since the zero solution is stable for τ=0, then it must follow that $\tau^+{}_0 < \tau^-{}_0$, since the multiplicity of roots with positive real parts cannot become negative. We observe that:

$$\tau^+{}_{j+1} - \tau^+{}_j = \frac{2\pi}{\omega_+} < \frac{2\pi}{\omega_-} = \tau^-{}_{j+1} - \tau^-{}_j .$$

Therefore, there can be only a finite number of switches between stability and instability. Moreover, it is easy to see that there exist values of the parameters that realize any number of such stability switches. In our case, there exists a value of τ, $\tau = \tau^*$ such that at $\tau = \tau^*$ a stability switch occurs from stable to unstable, and for $\tau > \tau^*$ the solution remains unstable. So that the zero solution can either be unstable for τ>0, or exhibit stability switches. As τ increased, the multiplicity of roots for Reλ>0 is increased by two whenever τ passes through a value of $\tau_j{}^+$ and it is decreased by two whenever τ passes through a value of $\tau_j{}^-$. Now, since the zero solution is stable for τ=0, k-switches from stability to instability to stability occur when the parameters are such

$$\tau_0{}^+ < \tau_0{}^- < \tau_1{}^+ < \tau_1{}^- < ... < \tau^+{}_{k-1}$$
$$< \tau^-{}_{k-1} < \tau^+{}_k < \tau^-{}_k ... .$$

This show that there exists a Hopf Bifurcation which takes place at $\tau = \tau^+{}_j$ and it is known that delay induced periodic oscillations. It follows that the linear stability of the equilibrium (h^*, m^*) is lost as the delay in response increases and when $\tau = \tau^+{}_j$.

From the above discussion, we have the following stability result of the positive equilibrium point of the system (1.2) and the existence of Hopf Bifurcations.

Theorem 3.1. *Assume that* $\alpha\beta > \gamma\delta$ *and* $m^* + h^* = 1$ *and* $\tau = \sigma = \tau$.

(a). If τ=0, the equilibrium point (h^*, m^*) *is locally asymptotically stable.*

(b). If (h_1) holds then the equilibrium point (h^*, m^*) *is locally asymptotically stable for* $\tau < \tau^+{}_0$.

(c). If (h_2) holds then the stability of the equilibrium (h^*, m^*) *can change a finite number of times as τ is increased, and eventually it becomes unstable.*

Remark 3.2. The statement on (c) of Theorem 3.1 imply that when $m^* + h^* = 1$ and τ=σ=τ the system (1.2) undergoes a Hopf Bifurcation at the positive equilibrium point (h^*, m^*) when $\tau = \tau^+{}_j$, for j=0,1,2... .

Next, we consider the Case (2). In this case upon simplification, the characteristic equation (3.4) can be rewritten as:

$$H(\lambda) = (\lambda + \gamma + \alpha m^* e^{-\lambda\tau})(\lambda + \delta + \beta h^* e^{-\lambda\sigma}) = 0. \text{De}$$

noting the two factors of H(λ) by $H_1(\lambda)$ and $H_2(\lambda)$ respectively, we have:

$$H_1(\lambda) = (\lambda + \gamma + \alpha m^* e^{-\lambda\tau}),$$
$$H_2(\lambda) = (\lambda + \delta + \beta h^* e^{-\lambda\sigma}).$$

Now, the stability investigation, for discrete delays is concerned with establishing that one of the characteristic equations $H_1(\lambda)=0$ and $H_2(\lambda)=0$ has infinite numbers of zeros that have no positive real parts. Suppose $\lambda = i\omega$ with w real and $\omega > 0$. Then at lest one of

$$i\omega + \gamma + \alpha m^* e^{-iw\tau} = 0 \;\; or \;\; iw + \delta + \beta h^* e^{-iw\sigma} = 0,$$

holds. We suppose that the first is satisfied and denoting

$$P(\lambda) = \lambda + \gamma \;\; and \;\; Q(\lambda) = \alpha m^* .$$

One can easily see that P(0)+Q(0)≠0, and

$$F_1(\omega) = \omega^2 + (\alpha m^*)^2 - \gamma^2 = 0.$$

It is clear that $F_1(\omega)$ has a real positive root if and only if

$$\lambda > \alpha m^* = \frac{\alpha\beta - \gamma\delta}{\beta + \delta} \Rightarrow \gamma > \frac{\alpha\beta}{2\delta + \beta},$$

and then the positive equilibrium point (h^*, m^*) is locally asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, where $\tau_0 = \theta_1 / \omega_1$, and

$$\omega_1 = \left(\gamma^2 - \left(\frac{\alpha\beta}{2\delta + \beta} \right)^2 \right)^2, \quad \theta_1 = arccot(\frac{\gamma}{\omega_1}).$$

Also, we can easily see that if the second case hold, and if

$$\gamma > (1/2)\left(\left(\alpha(\beta - \delta)\right)/\delta\right),$$

then the positive equilibrium point (h^*, m^*) is locally asymptotically stable when $\sigma < \sigma_0$ and unstable when $\sigma > \sigma_0$, where $\sigma_0 = \{\theta_2 / \omega_2\}$ and

$$\omega_2 = \left(\delta^2 - \left(\frac{\alpha\beta}{2\gamma + \alpha} \right)^2 \right)^2, \quad \theta_2 = arccot(\frac{\delta}{\omega_2}).$$

From above discussion we have the following stability properties of the positive equilibrium point (h^*, m^*) in the case when $m^* + h^* - 1 = h^* m^*$.

Theorem 3.2. *Assume that* $\alpha\beta > \gamma\delta$, *and* $m^* + h^* - 1 = h^* m^*$. *If*

$$\gamma > \max\left(\left(\frac{\alpha\beta}{2\delta + \beta} \right), \frac{1}{2}\left(\frac{\alpha(\beta - \delta)}{\delta} \right) \right),$$

then the positive equilibrium point (h^*, m^*) *is locally asymptotically stable when* $\tau < \tau_c$ *and unstable when* $\tau > \tau_c$, *where* $\tau_c = max\{\sigma_0, \tau_0\}$, *and* τ_0 *and* σ_0 *be as defined above.*

Remark 3.3. From Theorem 3.2, we see that when $m^* + h^* - 1 = h^* m^*$ the system (1.2) undergoes a Hopf Bifurcation at the positive equilibrium point (h^*, m^*) when $\tau = \tau_c$. In the case when $\gamma = \alpha\beta / (2\delta + \beta)$ and $\delta = \alpha\beta / (2\gamma + \alpha)$, we see that there is a zero solution of $F_1(\omega) = 0$ and $F_2(\omega) = 0$. However λ is not a root of $H_1(\lambda) = 0$ and $H_2(\lambda) = 0$. It remains to study the stability switches when $\gamma = \alpha\beta / (2\delta + \beta)$ and $\delta = \alpha\beta / (2\gamma + \alpha)$, $\tau \neq \sigma$ and $m^* + h^* - 1 \neq h^* m^*$

Remark 3.4. For the trivial equilibrium $E_0 = (h_0, m_0) = (0, 0)$, which corresponds to the free disease case i. e., when $\gamma\delta = \alpha\beta$, the corresponding characteristic equation is given by:

$$\lambda^2 + \lambda A + B - E_1 e^{-\lambda(\tau + \sigma)} = 0, \qquad (3.7)$$

where $A = \delta + \gamma$, $B = \gamma\delta$ and $E_1 = \alpha\beta$. We note that in the case when $\tau + \sigma = 0$, we see that (3.7) has a negative root so that the fixed point E_0 is asymptotically stable. If iw ($w > 0$) is a root of (3.7), then as above we see that w satisfies the equation

$$u^2 - (2B - A^2)u = 0, \text{ where } \omega^2 = u.$$

The last equation has no positive roots since $2B - A^2 = 2\gamma\delta - (\delta + \gamma)^2 < 0$. This implies that there are no stability switches as $\tau + \sigma$ increases.

Remark 3.5. In Theorems 3.1 and 3.2 according to the delays we found that there exists a family of periodic solutions bifurcate from the steady state E_* at the critical value of $\tau = \tau^+_j$ (for j=0, 1,...) provided that the condition (h_2) holds. It would be interesting to study the direction, stability and period of these periodic solutions bifurcating from the positive steady state. This question is open and the interesting reader can use the idea of), Hassard, B., Kazarinoff, D. and Wan,Y. 1981 and derive the explicit formulas determining these factors at the critical value of the delays using the normal form and the center manifold theory.

Lyapunov Functional Method

In this Section, we establish some sufficient conditions for local asymptotic stability of the equilibrium point when $\tau \neq \sigma$. Our strategy in the proof will be employed in a straightforward manner by constructing a suitable function and prove that this function is a Lyapunov functional of the system (3.3).

Consider the autonomous system of the delay differential equation (1.5) such that F(0)=0 and

$$F : C([-\tau^*,0],R^2) \to R^2,\ \tau^* > 0,$$

is Lipshitzian, where $C([-\tau^*,0], R^2)$ is the set of continuous functions defined on $[-\tau^*,0]$ with the norm

$$\|\phi\| = max_{\theta\in[-\tau^*,0]} |\phi(\theta)|,$$

where |.| is any norm in R^2. The following lemma is adapted from Kuang, Y. 1991, [Kuang, Y. 1993, Corollary 5.2] and will be useful in the proof of the main result in this section.

Lemma 4.1 *[Kuang Y. 1993]. Let* $w_1(.)$ *and* $w_2(.)$ *be nonnegative continuous scalar functions such that* $w_i(0)=0$, *i=1,2;* $lim_{t\to\infty} w_1(t) = +\infty$, $w_2(t)>0$ *for* $t>0$. *Let* $V{:}C\to R$ *be continuously differentiable scalar functional and S a nonempty subset of C for which the following are satisfied*

$$V(\varphi) \ge w_1(|\varphi(0)|),$$
$$|V'(\varphi)|_{(1.5)} \le -w_2(|\varphi(0)|).$$

Then X=0 is asymptotically stable with respect to the set S. That is, solutions that stay in S converge to X=0.

To simplify the statement of the theorem, the following notations will be used:

$$a_{11} = \alpha m^*, a_{12} = \alpha(1-h^*),$$
$$a_{22} = \beta h^*, a_{21} = \beta(1-m^*),$$
$$r_{11} = a_{11} + \gamma, r_{22} = a_{22} + \delta,$$
$$A_{11} = \tau[2a_{11}r_{11} + a_{12}r_{11} + a_{12}a_{11}],$$
$$A_{12} = \tau a_{12}[2a_{12} + a_{11} + r_{11}],$$
$$A_{22} = \sigma[2a_{22}r_{22} + a_{21}r_{22} + a_{21}a_{22}],$$
$$A_{21} = \sigma a_{21}[2a_{21} + a_{22} + r_{22}],$$
$$A_1 = (\tau+\sigma)[a_{11}a_{21} + (1/2)a_{11}a_{22} + (1/2)a_{12}a_{21}],$$
$$A_2 = (\tau+\sigma)[a_{12}a_{22} + (1/2)a_{11}a_{22} + (1/2)a_{12}a_{21}].$$

Using the parameters in our model these values become

$$a_{11} = \alpha\big((\alpha\beta-\gamma\delta)/(\alpha(\beta+\delta))\big),$$
$$a_{12} = \alpha\big(1-\big((\alpha\beta-\gamma\delta)/((\alpha+\gamma)\beta)\big)\big),$$
$$a_{22} = \beta\big((\alpha\beta-\gamma\delta)/((\alpha+\gamma)\beta)\big),$$
$$a_{21} = \beta\big(1-\big((\alpha\beta-\gamma\delta)/(\alpha(\beta+\delta))\big)\big),$$
$$r_{11} = \alpha\big((\alpha\beta-\gamma\delta)/(\alpha(\beta+\delta))\big)+\gamma,$$
$$r_{22} = \beta\big((\alpha\beta-\gamma\delta)/((\alpha+\gamma)\beta)\big)+\delta,$$

$$A_{12} = \tau(\alpha/\beta)(\gamma/(\alpha+\gamma))(\beta+\delta)\times\begin{bmatrix} 2\alpha\big(1-\big((\alpha\beta-\gamma\delta)/((\alpha+\gamma)\beta)\big)\big) \\ +\alpha\big((\alpha\beta-\gamma\delta)/(\alpha(\beta+\delta))\big) \\ +\alpha\big((\alpha\beta-\gamma\delta)/(\alpha(\beta+\delta))\big)+\gamma \end{bmatrix},$$

$$A_{21} = \sigma(1/\alpha)\beta\delta\left(\frac{\alpha+\gamma}{\beta+\delta}\right)\times\begin{bmatrix} 2\beta\left(1-\left(\dfrac{\alpha\beta-\gamma\delta}{\alpha(\beta+\delta)}\right)\right) \\ +\beta\left(\dfrac{(\alpha\beta-\gamma\delta)}{(\alpha+\gamma)\beta}\right)+\beta\left(\dfrac{\alpha\beta-\gamma\delta}{(\alpha+\gamma)\beta}\right)+\delta \end{bmatrix}$$

$$A_1 = (\tau+\sigma)\begin{pmatrix} \dfrac{1}{2}\gamma\delta + \dfrac{(\alpha\beta-\gamma\delta)^2}{(2(\alpha+\gamma)(\beta+\delta))} \\ +\left(\dfrac{\beta}{\alpha}\right)\left(\dfrac{\delta(\alpha+\gamma)(\alpha\beta-\gamma\delta)}{(\beta+\delta)^2}\right) \end{pmatrix},$$

$$A_2 = (\tau+\sigma)\begin{pmatrix} \dfrac{1}{2}\gamma\delta + \left(\dfrac{(\alpha\beta-\gamma\delta)^2}{2(\alpha+\gamma)(\beta+\delta)}\right) \\ +\left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\gamma(\beta+\delta)(\alpha\beta-\gamma\delta)}{(\alpha+\gamma)^2}\right) \end{pmatrix},$$

$$A_{11} = \tau\begin{bmatrix} \left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\gamma(\alpha\beta-\gamma\delta)}{(\alpha+\gamma)}\right) \\ +\alpha\gamma + 2\beta\left(\dfrac{\alpha+\gamma}{(\beta+\delta)^2}\right)(\alpha\beta-\gamma\delta) \end{bmatrix},$$

$$A_{22} = \sigma\begin{bmatrix} (\beta/\alpha)(\delta/(\beta+\delta))(\alpha\beta-\gamma\delta)+\beta\delta \\ +2\alpha\big(((\beta+\delta))/((\alpha+\gamma)^2)\big)(\alpha\beta-\gamma\delta) \end{bmatrix}.$$

Now, we state and prove the main result in this section.

Theorem 4.1. *Assume that $\alpha\beta > \gamma\delta$. If*

(H_1). $r_{22} + r_{11} \geq 2[a_{12} + a_{21}]$,

(H_2). $\left(\frac{3}{2}\right) r_{11} + a_{21} - \frac{r_{22}}{2} > A_{11} + A_{21} + A_1$

(H_3). $\frac{3}{2} r_{22} + a_{12} - \frac{r_{11}}{2} > A_{22} + A_{12} + A_2$.

Then the positive equilibrium point (h^*, m^*) is locally asymptotically stable.

Proof. From Theorem 3.1, we note that the stability of the positive equilibrium (h^*, m^*) of system (1.2) follows form the stability of the zero solution of (3.3). The equations in (3.3) can be rewritten as

$$\begin{aligned} &\frac{d}{dt}[x(t) - a_{11}\int_{t-\tau}^{t} x(s)ds - a_{12}\int_{t-\tau}^{t} y(s)ds] \\ &= -r_{11}x(t) - a_{12}y(t), \\ &\frac{d}{dt}[y(t) - a_{22}\int_{t-\sigma}^{t} y(s)ds - a_{21}\int_{t-\sigma}^{t} x(s)ds] \\ &= -r_{22}y(t) - a_{21}y(t), \end{aligned} \tag{4.1}$$

We denote

$$\begin{aligned} X(t) &= [x(t) - a_{11}\int_{t-\tau}^{t} x(s)ds - a_{12}\int_{t-\tau}^{t} y(s)ds] \\ Y(t) &= [y(t) - a_{22}\int_{t-\sigma}^{t} y(s)ds - a_{21}\int_{t-\sigma}^{t} x(s)ds]. \end{aligned} \tag{4.2}$$

Let

$$V_1(t) := X^2(t) + W_1(t), \tag{4.3}$$

where

$$W_1(t) := (r_{11} + a_{12}) \times [a_{11}\int_{t-\tau}^{t}\int_{s}^{t} x^2(z)dzds + a_{12}\int_{t-\tau}^{t}\int_{s}^{t} y^2(z)dzds],$$

$$V_2(t) := Y^2(t) + W_2(t), \tag{4.4}$$

where

$$W_2(t) := (r_{22} + a_{21}) \times [a_{22}\int_{t-\sigma}^{t}\int_{s}^{t} y^2(z)dzds + a_{21}\int_{t-\sigma}^{t}\int_{s}^{t} x^2(z)dzds],$$

and let

$$V_3(t) := -X(t)Y(t) + W_3(t), \tag{4.5}$$

Where

$$\begin{aligned} W_3(t) &:= \frac{1}{2}(a_{11} + a_{12}) \\ &\times [a_{21}\int_{t-\sigma}^{t}\int_{s}^{t} x^2(z)dzds + a_{21}\int_{t-\sigma}^{t}\int_{s}^{t} y^2(z)dzds] \\ &+ (a_{21} + a_{22}) \\ &\times [a_{11}\int_{t-\tau}^{t}\int_{s}^{t} x^2(z)dzds + a_{12}\int_{t-\tau}^{t}\int_{s}^{t} y^2(z)dzds], \end{aligned}$$

We define the function V(t) by

$$V(t) := V_1(t) + V_2(t) + V_3(t). \tag{4.6}$$

Now, we prove that the function V(t) is a Lyapunov functional for the system (4.1). From the fact that $x^2+y^2 \geq 2xy > xy$, we see that $V(t) > 0$. Now, we calculate that derivative of the function V(t) along solutions of the system (4.1). We start with the derivative of $V_1(t)$. By calculating the derivative of $X^2(t)$ along solutions of (4.1), find that

$$\begin{aligned} \frac{dX^2(t)}{dt} &= -2r_{11}x^2(t) - 2a_{12}x(t)y(t) \\ &+ 2r_{11}a_{11}x(t)\int_{t-\tau}^{t} x(s)ds + 2r_{11}a_{12}x(t)\int_{t-\tau}^{t} y(s)ds \\ &+ 2a_{11}a_{12}y(t)\int_{t-\tau}^{t} x(s)ds \\ &+ 2a^2_{12}y(t)\int_{t-\tau}^{t} y(s)ds \end{aligned}$$

Using the inequality $a^2+b^2 \geq 2ab$, we have

$$\begin{aligned} \frac{dX^2(t)}{dt} &\leq -2r_{11}x^2(t) - 2a_{12}x(t)y(t) \\ &+ r_{11}a_{11}[\tau x^2(t) + \int_{t-\tau}^{t} x^2(s)ds] \\ &+ r_{11}a_{12}[\tau x^2(t) + \int_{t-\tau}^{t} y^2(s)ds] \\ &+ a_{11}a_{12}[\tau y^2(t) + \int_{t-\tau}^{t} x^2(s)ds] \\ &+ a^2_{12}[\tau y^2(t) + \int_{t-\tau}^{t} y^2(s)ds] \end{aligned}$$

$$\begin{aligned}&= -2r_{11}x^2(t) - 2a_{12}x(t)y(t) \\ &+a_{11}\tau[r_{11}x^2 + a_{12}y^2] \\ &+a_{12}\tau[r_{11}x^2(t) + a_{12}y^2(t)] \\ &+a_{11}[r_{11} + a_{12}\int_{t-\tau}^{t} x^2(s)ds] \\ &+a_{12}[r_{11} + a_{12}\int_{t-\tau}^{t} y^2(s)ds].\end{aligned} \quad (4.7)$$

From the definition of $W_1(t)$, we find that:

$$\begin{aligned}\frac{d}{dt}W_1(t) &= (r_{11} + a_{12})a_{11} \\ &\times[\tau x^2(t) - \int_{t-\tau}^{t} x^2(s)ds] \\ &+(r_{11} + a_{12})a_{12}[\tau y^2 - \int_{t-\tau}^{t} y^2(s)ds].\end{aligned} \quad (4.8)$$

Then form (4.7) and (4.8), we get

$$\begin{aligned}\frac{d}{dt}V_1(t) &\le -2r_{11}x^2(t) - 2a_{12}x(t)y(t) \\ &+a_{11}\tau[r_{11}x^2(t) + a_{12}y^2(t)] \\ &+a_{12}\tau[r_{11}x^2(t) + a_{12}y^2(t)] \\ &+(r_{11} + a_{12})a_{11}\tau x^2(t) + (r_{11} + a_{12})a_{12}\tau y^2(t) \\ &= -[2r_{11} - 2a_{11}r_{11}\tau - a_{12}r_{11}\tau - a_{12}a_{11}\tau]x^2(t) \\ &-2a_{12}x(t)y(t) + \\ &+[2a_{12}^2\tau + a_{11}a_{12}\tau + r_{11}a_{12}\tau]y^2(t).\end{aligned}$$

and this implies that:

$$\begin{aligned}\frac{d}{dt}V_1(t) &\le -[2r_{11} - A_{11}]x^2(t) \\ &-a_{12}x(t)y(t) + A_{12}y^2(t).\end{aligned} \quad (4.9)$$

Now, we calculate the derivative of $V_2(t)$ along solutions of (4.1). By calculating the derivative of $Y^2(t)$ along the solutions of (4.1), we obtain after using the inequality $x^2+y^2\geq 2xy$,

$$\begin{aligned}\frac{dY^2(t)}{dt} &= -2r_{22}y^2(t) - 2a_{12}x(t)y(t) \\ &+a_{22}\sigma[r_{22}y^2 + a_{21}x^2] + a_{21}\sigma[r_{22}y^2(t) + a_{21}x^2(t)] \\ &+a_{22}[r_{22} + a_{21}\int_{t-\sigma}^{t} y^2(s)ds] + a_{21}[r_{22} + a_{21}\int_{t-\sigma}^{t} x^2(s)ds].\end{aligned} \quad (4.10)$$

From the definition of the function $W_2(t)$, we get:

$$\begin{aligned}\frac{d}{dt}W_2(t) &= (r_{22} + a_{21})a_{22}[\sigma y^2(t) - \int_{t-\sigma}^{t} y^2(s)ds] \\ &+(r_{22} + a_{21})a_{21}[\sigma x^2(t) - \int_{t-\sigma}^{t} x^2(s)ds].\end{aligned} \quad (4.11)$$

From (4.10) and (4.11), we find that:

$$\frac{d}{dt}V_2(t) \le -[2r_{22} - A_{22}]y^2 - 2a_{21}xy + A_{21}x^2. \quad (4.12)$$

Next, we calculate the derivative of the function $V_3(t)$. By calculating the derivative of $-X(t)Y(t)$ along solutions of (4.1) and the fact that $-r_{11}\leq -a_{11}$ and $-r_{22}\leq -a_{22}$, we have:

$$\begin{aligned}&\frac{d}{dt}(-X(t)Y(t)) \\ &\le a_{21}x^2(t) + a_{12}y^2(t) + (r_{22} + r_{11})x(t)y(t) \\ &-a_{11}a_{22}y(t)\int_{t-\tau}^{t} x(s)ds - a_{22}a_{12}y(t)\int_{t-\tau}^{t} y(s)ds \\ &-a_{21}a_{11}x(t)\int_{t-\tau}^{t} x(s)ds - a_{12}a_{21}x(t)\int_{t-\tau}^{t} y(s)ds \\ &-a_{11}a_{22}x(t)\int_{t-\sigma}^{t} y(s)ds - a_{11}a_{21}x(t)\int_{t-\sigma}^{t} y(s)ds \\ &-a_{12}a_{22}y(t)\int_{t-\sigma}^{t} y(s)ds - a_{12}a_{21}y(t)\int_{t-\sigma}^{t} x(s)ds.\end{aligned} \quad (4.13)$$

From the definition of the function $W_3(t)$, we have

$$\begin{aligned}&\frac{d}{dt}W_3(t) \\ &= \frac{1}{2}[(a_{11} + a_{12})a_{21}(\sigma x^2(t) - \int_{t-\sigma}^{t} x^2(s)ds] \\ &+\frac{1}{2}[(a_{11} + a_{12})a_{22}(\sigma y^2(t) - \int_{t-\sigma}^{t} y^2(s)ds] \\ &+\frac{1}{2}[(a_{21} + a_{22})a_{11}(\tau x^2(t) - \int_{t-\tau}^{t} x^2(s)ds] \\ &+\frac{1}{2}[(a_{21} + a_{22})a_{12}(\tau y^2(t) - \int_{t-\tau}^{t} y^2(s)ds].\end{aligned} \quad (4.14)$$

From (4.13) and (4.14), we have:

$$\frac{d}{dt}V_3 \le (a_{21}+A_1)x^2(t) +(a_{12}+A_2)y^2(t)+(r_{22}+r_{11})x(t)y(t). \quad (4.15)$$

Then from (4.9), (4.12) and (4.15), we find that:

$$\begin{aligned}\frac{d}{dt}V(t) \le & -[2r_{11}-A_{11}]x^2(t) \\ & -2a_{12}x(t)y(t)+A_{12}y^2(t) \\ & -[2r_{22}-A_{22}]y^2(t)-[2r_{22}-A_{22}]y^2(t) \\ & -a_{21}x(t)y(t)+A_{21}x^2(t) \\ & +(a_{21}+A_1)x^2(t)+(a_{12}+A_2)y^2(t) \\ & +(r_{22}+r_{11})x(t)y(t).\end{aligned}$$

From (H_1), and the fact that $x^2+y^2\ge 2xy$, we obtain

$$\begin{aligned}\frac{d}{dt}V(t) \le & -[2r_{11}-A_{11}-A_{21}-a_{21}-A_1]x^2(t) \\ & -[2r_{22}-A_{22}-A_{12}-a_{12}+A_2]y^2(t) \\ & +[(1/2)(r_{22}+r_{11})-(a_{12}+a_{21})]x^2(t) \\ & +[(1/2)(r_{22}+r_{11})-(a_{12}+a_{21})]y^2(t).\end{aligned}$$

This implies that:

$$\frac{d}{dt}V(t) \le -\zeta_1 x^2(t)-\zeta_2 y^2(t), \quad (4.16)$$

where

$$\zeta_1 = [(3/2)r_{11}+a_{12}-(1/2)r_{22} -A_{11}-A_{21}-A_1]-$$

$$\zeta_2 = [(3/2)r_{22}+a_{21}-(1/2)r_{11} -A_{22}-A_{12}-A_2].$$

The conditions (H_2) and (H_3) show that $\zeta_1>0$ and $\zeta_2>0$ and therefore $-\zeta_1x^2(t)-\zeta_2y^2(t)$ is negative definite. Now, since the solutions are bounded, Lemma 4.1 ensure that $\lim_{t\to\infty}x(t)=0$ and $\lim_{t\to\infty}y(t)=0$ and hence $\lim_{t\to\infty}h(t)=h^*$ and $\lim_{t\to}m(t)=m^*$. The proof is complete.

Discussion

In this paper, we consider the delay malaria epidemic model. We have established some sufficient conditions for stability of the equilibrium points. The main results are proved by two different methods. The first one is a complete analysis of the characteristic equations, when the delay are equal, of the corresponding linearized equations and proved that there exists stability switches and a there is a hopf bifurcations. In the case when the delays are different we used the Lyapunov method and established some sufficient conditions for local stability. The local stability means that the disease will persist if there is no immigration or emigration of the population.

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استقرار وتفريقات هوبف للنموذج الرياضي الذي به تأخير لوباء الملاريا

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ملخص البحث. توصف انتشار بعض الأمراض بمعادلات تفاضلية غير خطية بها تأخير وعادة ما يكون من الصعوبة إيجاد الحلول لهذه المعادلات وذلك لمعرفة سلوك الظاهرة الذي منه يمكن التنبؤ والتوقع لسلوك هذه الأمراض.
فى هذا البحث سوف يتم دراسة استقرار الحلول للنموذج الرياضي الذي يصف وباء الملاريا وكذلك تأثير فترات الحضانة على سلوك الحلول ومنها يمكن استنتاج أن هناك تفريقات هوبف وسوف نوضح أنه في حالة عدم وجود فترات حضانة للمرض لا يوجد هناك حلول دورية هذا البحث سوف نقوم بتطبيق طريقتين مختلفتين لدراسة استقرار الحلول واثبات أنه عند زيادة فقرات الحضانة يوجد تفريقات هويف ومنها يمكن معرفة وجود الحلول الدورية.