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On Expandable Spaces

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A new type of mapping having property (E) not closed but preserving the counterimages of expandable spaces is defined. A product theorem for expandable spaces is proved. Several sum theorems are obtained.

Krajewski (1971) introduced the notion of expandable spaces and obtained various related results. Smith and Krajewski (1971) characterized expandable spaces in terms of open A_σ covers.

In this article open A_σ covers are used to define a new type of mapping that has property (E), that is, not closed but preserving the counterimages of expandable spaces. Then, a product theorem is proved for expandable spaces. Also, several sum theorems are obtained. All spaces are assumed to be T_1 topological spaces.

To proceed we need the following definitions and facts:

Definition 1 (*Krajewski* 1971)

A space X is called expandable iff for every locally finite collection $\{F_\alpha | \alpha \in \Lambda\}$ of subsets of X there exists a locally finite open collection of subsets $\{G_\alpha | \alpha \in \Lambda\}$ such that for each $\alpha \in \Lambda$, $F_\alpha \subseteq G_\alpha$.

Definition 2 (*Burke* 1969)

A space X is called subparacompact iff every open cover has a σ-locally finite closed refinement.

Definition 3 (*Smith and Krajewski* 1971)

A cover $\underset{\sim}{U}$ of a space X is called A_σ cover iff $\underset{\sim}{U}$ has a σ-locally finite refinement (not necessarily open).

Theorem 1 (*Smith and Krajewski* 1971)

A space X is expandable iff every open A_σ cover has an open locally finite refinement.

Definition 4 (*Dieudonné* 1944)

A Hausdorff space X is called paracompact iff every open cover of X has an open locally finite refinement.

Theorem 2 (*Smith and Krajewski* 1971)

(i) Every paracompact space is expandable.
(ii) Every regular subparacompact expandable space is paracompact.

Definition 5 (*Katuta* 1967)

Let $\underset{\sim}{A}$ be a family of subsets of a space X well ordered by $<$. $\underset{\sim}{A}$ is called an order locally finite family iff for each $A \in \underset{\sim}{A}$; $\{A' | A' < A\}$ is locally finite at each point of A.

Definition 6

A mapping $f: X \to Y$ is said to have property (E) iff for each open A_σ cover $\underset{\sim}{U}$ of X there is an open A_σ cover $\underset{\sim}{V}$ of Y and an open refinement $\underset{\sim}{W}$ of $\underset{\sim}{U}$ such that for each V in $\underset{\sim}{V}$, $f^{-1}(V) \subset \bigcup\{W | W \in \underset{\sim}{W}_V\}$, where $\underset{\sim}{W}_V$ is locally finite and $\underset{\sim}{W}_V \subset \underset{\sim}{W}$.

Definition 7

A subset F of a space X is called expandable relative to X iff every open A_σ cover of F in X has an open locally finite refinement in X.

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Theorem 3

Let f be a continuous mapping from a space X onto an expandable space Y. Then X is expandable iff f has property (E).

The proof follows from the following two lemmas:

Lemma 1

Any continuous mapping defined on an expandable space has property (E).

Lemma 2

Let f be a continuous mapping from a space X onto an expandable space Y. Then X is expandable if f has property (E).

Proof of Lemma 2. Let $\underset{\sim}{U}$ be an open A_σ cover of X. Since X is expandable, $\underset{\sim}{U}$ has an open locally finite refinement $\underset{\sim}{W}$. Now $\{Y'\}$ is an open A_σ cover of Y such that $f^{-1}(Y)$ is contained in the union of members of $\underset{\sim}{W}$. Thus f has property (E).

Proof of Lemma 3. Let $\underset{\sim}{U}$ be an open A_σ cover of X. Since f has property (E), there is an open A_σ cover $\underset{\sim}{V}$ of Y and an open refinement $\underset{\sim}{W}$ of $\underset{\sim}{U}$ such that for each V in $\underset{\sim}{V}$, $f^{-1}(V)$ is contained in $\bigcup\{W|W \in \underset{\sim}{W}_V\}$, where $\underset{\sim}{W}_V$ is locally finite and $\underset{\sim}{W}_V \subset \underset{\sim}{W}$. Since Y is expandable, $\underset{\sim}{V}$ has an open locally finite refinement $\underset{\sim}{H}$. Let $S = \{f^{-1}(H) \cap W|H \in H, W \in W_V\}$. It is easy to see that S is an open locally finite refinement of $\underset{\sim}{U}$.

Theorem 4

Let M be a closed expandable subset of a space X. If F is closed in the interior G of M, then F is expandable relative to X.

Proof. Let $\underset{\sim}{U}$ be an open A_σ cover of F in X. Then $\underset{\sim}{B} = \{U \cap M|U \in \underset{\sim}{U}\} \cup \{M - F\}$ is an open A_σ cover of M. Since M is expandable, $\underset{\sim}{B}$ has an open locally finite refinement with respect to M. Let $\underset{\sim}{V}$ be the collection of members of this refinement such that each V in $\underset{\sim}{V}$ is contained in some member of $\{U \cap M|U \in U\}$. Let $\underset{\sim}{W} = \{V \cap G|V \in \underset{\sim}{V}\}$, then $\underset{\sim}{W}$ is an open locally finite refinement of $\underset{\sim}{U}$ in X.

Corollary 1

A closed subset of an expandable space X is expandable relative to X.

Theorem 5

Let f be a closed continuous mapping from a space X onto a regular subparacompact space Y. If for each y in Y, $f^{-1}(y)$ is expandable relative to X, then f has property (E).

Proof. Let $\underset{\sim}{U}$ be an open A_σ cover of X. Since $f^{-1}(y)$ is expandable relative to X, U has an open locally finite refinement in X, say, $\underset{\sim}{A}_y = \{A_\alpha|\alpha \in \Lambda_y\}$ and $\underset{\sim}{A}_y$ covers

$f^{-1}(y)$. Let $O_y = Y - f(X - \bigcup_{\alpha \in \Lambda_y} A_\alpha)$. Since f is closed, O_y is open in Y and $f^{-1}(O_y) \subset \bigcup_{\alpha \in \Lambda_y} A_\alpha$. Now $\underset{\sim}{A} = \{\underset{\sim}{A}_y | y \in Y\}$ is an open refinement of $\underset{\sim}{U}$. Since X is subparacompact, $\{O_y | y \in Y\}$ is an open A_σ cover of Y such that for each $y \in Y$, $f^{-1}(O_y)$ is contained in the union of members of $\underset{\sim}{A}_y$ and $\underset{\sim}{A}_y$ is a locally finite subfamily of $\underset{\sim}{A}$. Thus f has property (E).

Remark 1

The converse of the above theorem is not true. A counterexample can be obtained by taking the identity mapping f from the Sorgenfrey line onto the reals with the usual topology. Since the Sorgenfrey line is expandable, by lemma 2 f has property (E). However, f is not closed; indeed let $A = [0, 1)$ and then A is closed in the Sorgenfrey topology but $f(A) = [0, 1)$ is not closed in the usual topology of the reals.

Theorem 6

Let f be a closed continuous mapping from a space X onto a regular subparacompact expandable space Y. Then X is expandable iff for each y in Y, $f^{-1}(y)$ is expandable relative to X.

Proof. Suppose X is expandable. For each y in Y, $f^{-1}(y)$ is closed in X; hence by corollary 1, $f^{-1}(y)$ is expandable relative to X.

For the converse, let f be a closed continuous mapping from the space X onto the regular subparacompact space Y such that for each y in Y, $f^{-1}(y)$ is expandable relative to X. Then by theorem 5, f has property (E). Therefore, by theorem 3, X is expandable.

Theorem 7

For any closed set F of a space X, the following are equivalent:

(a) F is expandable relative to X.

(b) F is expandable and the boundary $\mathrm{Bd}(F)$ is expandable relative to X.

Proof. (a)$\Rightarrow$(b). It is clear that F is expandable. Let $\{U_\alpha | \alpha \in \Lambda\}$ be an open A_σ cover of $\mathrm{Bd}(F)$ in X. Let $V = F - \mathrm{Bd}\,(F)$; then there is an open set U in X such that $V = U \cap F$. Now $\underset{\sim}{B} = \{U\} \cup \{U_\alpha | \alpha \in \Lambda\}$ is an open A_σ cover of F in X. Therefore $\underset{\sim}{B}$ has a locally finite open refinement in X. Now it is easy to see that $\mathrm{Bd}(F)$ is expandable relative to X.

(b)$\Rightarrow$(a). Let $\underset{\sim}{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open A_σ cover of F in X. Since F is expandable there is a locally finite open (in F) refinement $\{W_\beta | \beta \in \Gamma\}$ of $\{F \cap U_\alpha | \alpha \in \Lambda\}$. For each β, let $W'_\beta = W_\beta \cap \mathrm{Int}\,(F)$. Since $\mathrm{Bd}\,(F)$ is expandable relative to X, $\underset{\sim}{U}$ has an open (in X) refinement $\underset{\sim}{H}$ which is locally finite in X and covers $\mathrm{Bd}\,(F)$.

Now, $\{W'_\beta|\beta \in \Gamma\} \cup \{H|H \in \underset{\sim}{H}\}$ is an open locally finite refinement of $\underset{\sim}{U}$ in X which covers F.

Theorem 8

Let f be a closed continuous mapping from a space X onto a regular subparacompact expandable space Y. Then X is expandable iff for each y in Y, $f^{-1}(y)$ is expandable and the boundary $\mathrm{Bd}\,(f^{-1}(y))$ is expandable relative to X.

Proof. Suppose X is expandable. For each y in Y, $f^{-1}(y)$ is closed in X, therefore by corollary 1, $f^{-1}(y)$ is expandable relative to X and hence by theorem 7, $f^{-1}(y)$ is expandable and the boundary $\mathrm{Bd}\,(f^{-1}(y))$ is expandable relative to X.

For the converse, let f be a closed continuous mapping from the space X onto the regular subparacompact space Y such that for each y in Y, $\mathrm{f}^{-1}(y)$ is expandable and the boundary $\mathrm{Bd}\,(f^{-1}(y))$ is expandable relative to X. Then, by theorem 7, we obtain that for each y in Y, $f^{-1}(y)$ is expandable relative to X. Hence by theorem 6, X is expandable.

Theorem 9 (*Krajewski* 1971)

Let X be an expandable space and Y be a compact space. Then $X \times X$ is expandable.

Theorem 10

Let X be a regular subparacompact expandable space and Y be an expandable space. Then $X \times Y$ is expandable iff for each $x \in X$ there is an open neighbourhood U_x of x such that $\bar{U}_x \times Y$ is expandable.

Proof. The "only if" part of the proof is obvious; for the "if" part it suffices to show that the projection $p: X \times Y \to X$ has property (E). Let $\underset{\sim}{U}$ be an open A_σ cover of $X \times Y$. For each $x \in X$ there is an open neighbourhood U_x of x such that $\bar{U}_x \times Y$ is expandable. Since X is regular there is an open neighbourhood V_x of x such that $V_x \times Y \subset \bar{V}_x \times Y \subset U_x \times Y \subset \bar{U}_x \times Y$. By theorem 4, $\bar{V}_x \times Y$ is expandable relative to X. Therefore $\underset{\sim}{U}$ has an open locally finite refinement in X, say, $\underset{\sim}{A}_x$ such that $\bar{V}_x \times Y \subset \bigcup \underset{\sim}{A}_x$. Now $\underset{\sim}{A} = \bigcup_{x \in X} \underset{\sim}{A}_x$ is an open refinement of $\underset{\sim}{U}$ and $\{V_x|x \in X\}$ is an open cover of X such that $p^{-1}(V_x)$ is contained in the union of a locally finite subfamily of $\underset{\sim}{A}$. Since X is subparacompact, $\{V_x|x \in X\}$ is an open A_σ cover of X. Thus, for each open A_σ cover $\underset{\sim}{U}$ of $X \times Y$ there is an open A_σ cover $\{V_x|x \in X\}$ of X and an open refinement $\underset{\sim}{A}$ of $\underset{\sim}{U}$ such that for each $x \in X$, $p^{-1}(V_x)$ is contained in the union of a locally finite subfamily of $\underset{\sim}{A}$. Hence, the projection $p: X \times Y \to X$ has property (E). By theorem 3, $X \times Y$ is expandable.

Corollary 2 (*Smith and Krajewski* 1971)

Let X be a paracompact locally compact space and Y be an expandable space. Then $X \times Y$ is expandable. The proof follows from theorems 2, 9, and 10.

Theorem 11

Let $\{F_\alpha | \alpha \in \Lambda\}$ be a locally finite cover of a space X such that for each $\alpha \in \Lambda$, F_α is expandable relative to X. Then X is expandable.

Proof. Let $\underset{\sim}{U}$ be an open A_σ cover of X. For each $\alpha \in \Lambda$, $\underset{\sim}{U}$ has an open (in X) locally finite (in X) refinement, say, $\underset{\sim}{A}^\alpha = \{A_\beta | \beta \in I^\alpha\}$ that covers F_α. Let $\underset{\sim}{V} = \{A_\beta | \beta \in I^\alpha, \alpha \in \Lambda\}$; then $\underset{\sim}{V}$ is a locally finite refinement of $\underset{\sim}{U}$. Indeed, let $x \in X$; then there is an open-boundary M_x of x such that $M_x \cap F_\alpha = \emptyset$ for all except finitely many indices, say, $\alpha_1, \alpha_2, \ldots, \alpha_n$; we can assume that $x \in F_{\alpha_i}$, $i = 1, 2, \ldots, n$. Since each one of the collections $\underset{\sim}{A}^{\alpha_1}, \ldots, \underset{\sim}{A}^{\alpha_n}$ is locally finite, for each $i = 1, 2, \ldots, n$ there is an open set W_i such that $x \in W_i$ and W_i intersects at most finitely many members of $\underset{\sim}{A}^{\alpha_i}$. Hence $W_1 \cap W_2 \cap \cdots \cap W_n \cap M_x$ is an open-boundary of x that intersects finitely many members of $\underset{\sim}{V}$.

Theorem 12

Let $\underset{\sim}{V} = \{V_\alpha | \alpha \in \Lambda\}$ be an order locally finite open cover of a space X such that the closure of each member of $\underset{\sim}{V}$ is expandable relative to X. Then X is expandable.

Proof. For each $\alpha \in \Lambda$, put $F_\alpha = \bar{V}_\alpha - \cup\{V_\beta | \beta < \alpha\}$ and $\underset{\sim}{F} = \{F_\alpha | \alpha \in \Lambda\}$. By a similar method used by Katuta (1967), one can show that $\underset{\sim}{F}$ is a locally finite closed cover of X. Now, for each $\alpha \in \Lambda$, F_α is expandable relative to X; therefore, by theorem 11, X is expandable.

Theorem 13

Let $\underset{\sim}{V} = \{V_\alpha | \alpha \in \Lambda\}$ be an order locally finite open cover of a space X such that for each $\alpha \in \Lambda$, $\bar{V}_\alpha$ is expandable and the boundary $\mathrm{Bd}(\bar{V}_\alpha)$ is expandable relative to X. Then X is expandable.

The proof follows from theorems 7 and 12.

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