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n-Compact and Scattered Spaces

H.Z. Hdeib
Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan

In this paper scattered and σ_n-scattered spaces are used to obtain some product theorems for n-compact spaces.

Some product theorems for n-compact spaces are obtained using scattered and σ_n-scattered spaces, where n stands for infinite cardinal number. For any family $\underset{\sim}{A}$, $|\underset{\sim}{A}|$ denotes the cardinal number of the indexing set of $\underset{\sim}{A}$. Furthermore, for any set B, $|B|$ will denote the cardinal number of B. We shall make use of the following definitions:

Definition 1

A subset A of a space X is called an F_n set iff A is the union of $\leqslant n$ closed subsets of X. The complement of an F_n set is called G_n set. A mapping $f: X \rightarrow Y$ is called a σ_n-closed mapping iff it maps closed sets onto F_n sets.

Observe that $F_{\aleph_0}$ sets are the F_σ sets and $G_{\aleph_0}$ sets are the G_δ sets. Therefore $\sigma_{\aleph_0}$-closed mappings will be called simply σ-closed mappings.

Definition 2 (*Ga'l* 1957)

A space X is called n compact iff every open cover of X has a subcover with cardinality $\leqslant n$.

Definition 3

Let X be a given space. The G_n-topology of X is the topology generated by the G_n-sets of X. The $G_{\aleph_0}$ topology will be called simply G_δ-topology.

Definition 4 (*Kuratowski* 1966, *p.* 78)

A space X is called scattered iff every nonempty closed subspace A has an isolated point $x \in A$.

Definition 5

A space X is called σ_n-scattered iff it is the union of $\leqslant n$ closed, scattered subspaces. σ_{ψ_0}-scattered spaces were first introduced by Nyikos (1977). He called them σ-scattered spaces.

σ_n-Closed Mappings

Clearly every closed mapping is σ_n closed. However, the converse is not true. Indeed, let X be a given space, and X^* be the set X provided with the G_n topology of X. Let $f: X^* \to X$ be the identity mapping. Then f is σ_n-closed but it is not closed.

Remark 1

Let $f: X \to Y$ be a continuous mapping from X onto Y; then f is a σ_n-closed mapping iff for each y in Y and any open set U such that $f^{-1}(y) \subset U$, there exists a G_n set O_y such that $y \in O_y$ and $f^{-1}(O_y) \subset U$. The proof is an easy consequence of the definition.

Theorem 1

Let X be an n-compact space and Y be any space. Then, the projection $p: X \times Y \to Y$ is a σ_n-closed mapping.

Proof. Let $y \in Y$ and U be any open set in $X \times Y$ such that $p^{-1}(y) = X \times \{y\} \subset U$. For each $(x, y) \in X \times \{y\}$, let O_x and $O_y(x)$ be open neighbourhoods of x and y such that $(x, y) \in O_x \times O_{y(x)} \subset U$. Now $\{O_x | x \in X\}$ is an open cover of X; therefore, it has a subcover $\{O_{x_\alpha} | \alpha \in \Lambda\}$ where $|\Lambda| \leqslant n$. Hence

$$X \times \{y\} \subset \bigcup_{\alpha \in \Lambda} O_{x_\alpha} \times O_y(x_\alpha) \subset U.$$

Let

$$O_y = \bigcap_{\alpha \in \Lambda} O_y(x_\alpha);$$

then

$$X \times \{y\} \subset \bigcup_{\alpha \in \Lambda} O_{x_\alpha} \times O_y \subset U$$

and O_y is a G_n set. Thus, for each y in Y there is a G_n set O_y such that $y \in O_y$ and $p^{-1}(O_y) \subset U$. Therefore by Remark 1, p is a σ_n-closed mapping.

Theorem 2

Let f be a σ_n-closed continuous mapping of a space X onto a space Y such that for each y in Y, $f^{-1}(y)$ is n compact. Then X is n compact if the G_n-topology of Y is so.

Proof. Let $\underset{\sim}{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open cover of X. Since $f^{-1}(y)$ is n compact, $f^{-1}(y) \subset \bigcup \{U_\alpha | \alpha \in \Lambda_y\}$, where $|\Lambda_y| \leq n$. Let

$$O_y = Y - f\Big(X - \bigcup_{\alpha \in \Lambda y} U_\alpha\Big).$$

Since f is σ_n closed, O_y is a G_n set. Now $\underset{\sim}{O} = \{O_y | y \in Y\}$ is a cover of Y consisting of G_n sets such that for each y in Y, $f^{-1}(O_y) \subset \bigcup_{\alpha \in \Lambda y} U_\alpha$ where $|\Lambda_y| \leq n$. Since the G_n-topology of Y is n compact, $\underset{\sim}{O}$ has a subcover of cardinality $\leq n$. Now X = union of $\leq n$ members of $f^{-1}(\underset{\sim}{O})$ and since each member of $f^{-1}(\underset{\sim}{O})$ is contained in the union of $\leq n$ members of $\underset{\sim}{U}$, it follows that X = union of $\leq n$ members of $\underset{\sim}{U}$. Hence X is n compact.

Products of *n*-Compact Spaces

In this section we shall prove some product theorems for n-compact spaces. For this purpose we need the following result:

Theorem 3

Let X be an n-compact scattered space. Then the G_δ-topology of X is n compact.

Proof. Let $\underset{\sim}{U}$ be an open cover of X by G_δ sets. Let $A = \{x \in X | x \in U$ and U is open $\Rightarrow U$ cannot be covered by the union of $\leq n$ members of $\underset{\sim}{U}\}$. Obviously A is closed. Assume that $A \neq \emptyset$. Since X is scattered A has an isolated point x, so there is an open set $V \subseteq X$ such that $V \cap A = \{x\}$. Choose $U^* \in U$ such that $x \in U^*$. Without loss of generality we can assume that $U^* = \bigcap \{V_i | i = 1, 2, \ldots\}$, where V_i is open for each $i = 1, 2, \ldots$ and $\bar{V}_{i+1} \subseteq V_i$. For each $i = 1, 2, \ldots, (\bar{V}_i - V_{i+1}) \subseteq X - A$; therefore each $y \in (\bar{V}_i - V_{i+1})$ has an open set M_y that can be covered by the union of $\leq n$ members of $\underset{\sim}{U}$. Since $\bar{V}_i - V_{i+1}$ is n compact, the collection $\{M_y | y \in \bar{V}_i - V_{i+1}\}$ has a subcover of cardinality $\leq n$. Hence each $\bar{V}_i - V_{i+1}$ can be covered by a subcollection $\underset{\sim}{U}_i$ of $\underset{\sim}{U}$ where $|\underset{\sim}{U}_i| \leq n$. Let $\underset{\sim}{H} = \{U | U \in \underset{\sim}{U}_i,\ i = 1, 2, \ldots\} \cup \{U^*\}$; then $\underset{\sim}{H}$ is a subcollection of $\underset{\sim}{U}$ that covers V and $|\underset{\sim}{H}| \leq n$. But this contradicts the fact that $x \in A$. Hence $A = \emptyset$. Therefore for each $x \in X$ there is an open set G_x such that G_x can be covered by the union of $\leq n$ members of $\underset{\sim}{U}$. Since X is n compact and $\underset{\sim}{G} = \{G_x | x \in X\}$ is an open cover of X, $\underset{\sim}{G}$ has a subcover $\underset{\sim}{G}^*$, where $|\underset{\sim}{G}^*| \leq n$. Now each member of $\underset{\sim}{G}^*$ can be covered by the union of $\leq n$ members of $\underset{\sim}{U}$. Hence $\underset{\sim}{U}$ has a subcover of cardinality $\leq n$.

Corollary 1

Let X be an n-compact, σ_n-scattered space. Then the G_δ-topology of X is an n-compact.

Proof. Let $\underset{\sim}{U}$ be an open cover of X by G_δ sets. Let $X = \bigcup_{\alpha \in \Lambda} F_\alpha$, where $|\Lambda| \leq n$ and F_α is a closed scattered subspace for each $\alpha \in \Lambda$. By theorem 3, for each $\alpha \in \Lambda$, the G_δ-subspace topology of F_α is n compact. Hence $\underset{\sim}{U}$ has a subcover $\underset{\sim}{U}_\alpha$ such that $|\underset{\sim}{U}_\alpha| \leq n$ and $\underset{\sim}{U}_\alpha$ covers F_α. Now $\underset{\sim}{U}^* = \bigcup \{\underset{\sim}{U}_\alpha | \alpha \in \Lambda\}$ is a subcover of $\underset{\sim}{U}$ that covers X and $|\underset{\sim}{U}^*| \leq n$.

Theorem 4

Let X be a Lindelöf space and Y be an n-compact, scattered space. Then $X \times Y$ is n compact.

Proof. By theorem 1 the projection $p: X \times Y \to Y$ is a σ-closed mapping. The rest of the proof follows from theorems 2 and 3.

Corollary 2

Let X be a Lindelöf space and Y be an n-compact σ_n-scattered space. Then $X \times Y$ is n compact.

The proof follows from theorem 4.

Theorem 5

Let X and Y be n-compact spaces. Let K be a compact subset of X such that $F \times Y$ is n compact for every closed set $F \subseteq X - K$. Then $X \times Y$ is n compact.

Proof. Let $\underset{\sim}{A}$ be an open cover of $X \times Y$. For each y in Y, $K \times \{y\}$ is compact. Thus $K \times \{y\} \subset \cup \underset{\sim}{A}_y$, where $\underset{\sim}{A}_y$ is a finite subcollection of A. Let U_y be an open set in X and V_y be an open set in Y such that $K \times \{y\} \subset U_y \times V_y \subset \cup \underset{\sim}{A}_y$. Now $\{V_y | y \in Y\}$ is an open cover of Y, so it has a subcover $\{V_{y_\alpha} | \alpha \in \Lambda\}$, where $|\Lambda| \leq n$. Since $\{U_{y_\alpha} \times V_{y_\alpha} | \alpha \in \Lambda\}$ is an open cover of $K \times Y$, it follows that $\bigcup \{\underset{\sim}{A}_{y_\alpha} | \alpha \in \Lambda\}$ is an open cover of $K \times Y$. But $(X - U_{y_\alpha}) \times Y$ is n compact, therefore $\underset{\sim}{A}$ has a subcover $\underset{\sim}{A}_\alpha$ that covers $(X - U_{y_\alpha}) \times Y$ and $|\underset{\sim}{A}_\alpha| \leq n$. Let $\underset{\sim}{S} = [\bigcup \{\underset{\sim}{A}_{y_\alpha} | \alpha \in \Lambda\}] \cup [\bigcup \{\underset{\sim}{A}_\alpha | \alpha \in \Lambda\}]$. Then S is a subcover of $\underset{\sim}{A}$ and $|\underset{\sim}{S}| \leq n$.

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الفضاءات المحكمة من النمط ن والفضاءات المشتتة

حسن زارع هديب
قسم الرياضيات ـ كلية العلوم ـ جامعة اليرموك ـ اربد ـ الاردن

في هذا البحث استخدمت الفضاءات المشتتة في اثبات مبرهنات حول جداءات الفضاءات المحكمة من النمط ن .