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Unified Approximate Tracking Control of Linear Systems with Unacceptable Zeros

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Abstract. This paper addresses the problem of modeling and control of linear continuous-time systems with unacceptable zeros in a sense that they are on the right-half plane of the S-domain (non-minimum phase) or cause the controller to have undesired behavior. The unified approximation of output tracking control using *output redefinition* will result in approximate output tracking, yet insuring stable internal dynamics of the system. Four different types of output redefinition are presented in a unified way and shown to satisfy different forms of control objectives through a simulation example of small-signal model of boost converter.

Introduction

In many practical applications, the control objective consists of output tracking with internal stability. A natural way to design such controllers is to use the pole-zero cancellation. To be able to meet this objective, one must assume that the system is minimum phase, that is all zeros of the transfer function are stable. It is obvious that the closed loop poles must be stable too. Even if all the zeros are located in the left half plane, some zeros, e.g. those on the real axis and close to $-\infty$, may cause the control signal to be highly oscillatory. To control such a system, it is not practical to try to perfectly track a desired output trajectory. Instead, one should be satisfied with approximate output tracking objective while insuring the stability of internal dynamics.

In this paper, the control problem of non-minimum phase systems will be considered. The output redefinition method for modeling and control will be discussed. The preliminary results of this paper were published in [1]. Four different versions of this technique will be presented in a unified way. They can be used to satisfy different control objectives. The first is the simplest and is known to introduce both magnitude and phase errors at all frequencies except dc [2]. The second was introduced for digital control in [3] and is known to introduce only magnitude error at all frequencies except dc. Another output redefinition method is derived and used to cancel the magnitude error at all frequencies. The second method can be modified in such a way that both the magnitude and phase errors at a given desired frequency is zero, provided that the desired output trajectory is a pure sinusoidal with zero dc component. Frequency-dependent version of this method is derived and shown to exhibit no phase error and no magnitude error if the output frequency is known in advance and the controller is designed accordingly. The main contribution of this paper is in the introduction of the zero magnitude error tracking controller and the modification of the zero phase error tracking controller to perfectly track an output with certain requirements. Another contribution is in presenting these output redefinition methods and corresponding controllers in a unified way that is easy and clear to understand and use. The controller design is presented in Section 3 followed by a simulation example in Section 4.

Output Redefinition

The paper considers the linear time-invariant system of the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

The (scalar) transfer function for the original small-signal model can be written as:

$$H(s) = C(sI - A)^{-1}B := \frac{N_a(s)N_u(s)}{D(s)} \tag{2}$$

where $N_a(s)$ polynomial includes the acceptable zeros and $N_u(s)$ polynomial includes all unacceptable zeros. Note that $N_u(s)$ must contain all zeros located on the right-half plane of the S-domain. Also, suppose that the relative degree of the system is r and the polynomial $N_u(s)$ is of order m, i.e. there are m unacceptable zeros. In the case where $N_u(s)$ is not unity, any controller based on direct inversion will not be practical since it generates unbounded or highly oscillatory internal dynamics. In this paper, a method based on output redefinition is introduced as a solution to this problem. More than one alternative for output redefinition are considered here to satisfy different control objectives.

Output redefinition relies on the fact that the approximate transfer function $\hat{H}(s)$ may be written in the form:

$$\hat{H}(s) = \frac{\hat{C}\ adj(sI - A)B}{det(sI - A)} \tag{3}$$

By defining $a_1, \ldots, a_n$ as the characteristic polynomial coefficients of the $n \times n$ matrix A:

$$det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n \tag{4}$$

it is possible to establish an explicit expression for the adjoint matrix of $sI - A$, namely:

$$adj(sI - A) = (I)s^{n-1} + (A + a_1 I)s^{n-2} + \cdots + (A^{n-1} + a_1 A^{n-2} + + a_{n-1} I) \tag{5}$$

Hence, it follows that the triple $(\mathrm{A}, \mathrm{B}, \hat{\mathrm{C}})$ will generate the approximate transfer function $\hat{\mathrm{H}}(\mathrm{s})$ provided that $\hat{\mathrm{C}}$ is computed according to:

$$\begin{bmatrix} (B)^T \\ ((A + a_1 I)B)^T \\ \vdots \\ ((A^{n-1} + a_1 A^{n-2} + + a_{n-1} I)B)^T \end{bmatrix} \hat{C} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{6}$$

where the new polynomial $\beta_1 s^n + \beta_2 s^{n-1} + \cdots + \beta_n$ is user defined and can be anything. This result makes use of the fact that two polynomials are identical if and only if their orderd coefficients are equal. It will prove useful later to define the two vectors $\hat{C}_1$ and $\hat{C}_2$ as follows:

$$\begin{aligned} \hat{C}_1 adj(sI - A)B &:= N_a(s) N_u(0) \\ \hat{C}_2 adj(sI - A)B &:= N_a(s) N_u(-s) \end{aligned} \tag{7}$$

A general description of the unified linear approximate dynamic model is given by the following:

$$\begin{aligned} \dot{x}_a(t) &= A_a x_a(t) + B_a d(t) \\ \hat{y}(t) &= C_a x_a(t) \end{aligned} \tag{8}$$

where the constant matrices (A_a, B_a, C_a) and the new state vector (x_a) are defined in different ways depending on the output redefinition method being used. This is the subject of the next section.

ZDCETC Output Redefinition

An intuitive way to solve the problem of unacceptable zeros is to replace $N_u(s)$ by its dc value, $N_u(0)$ and to assume that $\hat{y}(t)$ is a good approximation of $y(t)$. This approximate model will exhibit both magnitude and phase errors at all frequencies other than dc, hence the name *Zero DC Error Tracking Controller* (ZDCETC). The transfer function of such approximation is given by:

$$\hat{H}_1(s) = \frac{N_a(s)N_u(0)}{D(s)} := \hat{C}_a(sI - A_a)^{-1}B_a \tag{9}$$

where

$$\begin{aligned} A_a &:= A \\ B_a &:= B \\ C_a &:= \hat{C}_1 \\ x_a &:= x \end{aligned} \tag{10}$$

Note that the relative degree of this approximate model is $r_a = r + m$ since $N_u(s)$ is a non-zero polynomial of order m.

ZPETC Output Redefinition

If the control objective can not tolerate any phase error in the tracking process, another output redefinition method called *Zero Phase Error Tracking Controller (ZPETC)* **Error! Reference source not found** is to be used. This approximate model will have zero phase error at all frequencies. On the other hand, it will exhibit magnitude error at all frequencies other than dc. The transfer function for such approximation is given by:

$$\hat{H}_2(s) = \frac{N_a(s)N_u^2(0)}{D(s)N_u(-s)} := C_a(sI - A_a)^{-1}B_a \tag{11}$$

where

$$
\begin{aligned}
A_a &:= \begin{bmatrix} A & 0_{n\times m} \\ 0_{m-1\times n} & M_{m-1\times m} \\ \frac{k_0}{k_m}\hat{C}_1 & -\frac{k_0}{k_m} \cdots -\frac{k_{m-1}}{k_m} \end{bmatrix} \\
B_a &:= \begin{bmatrix} B \\ 0_{m\times 1} \end{bmatrix} \\
C_a &:= \begin{bmatrix} 0_{n\times 1} \\ 1 \\ 0_{m-1\times 1} \end{bmatrix} \\
x_a &:= \begin{bmatrix} x \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}
\end{aligned}
\tag{12}
$$

and

$$N_u(-s) = k_m s^m + k_{m-1} s^{m-1} + \cdots + k_1 s + k_0 \tag{13}$$

$$M_{m-1\times m}(i,j) = \begin{cases} 1 & j = i+1 \\ 0 & otherwise \end{cases} \tag{14}$$

By construction, $H(s)$ and $\hat{H}(s)$ are identical at zero frequency. Moreover, they have the same phase for all other frequencies. The relative degree of this approximate model is $r_a = r + 2m$.

ZMETC Output Redefinition

If the control objective can not tolerate any magnitude error in the tracking process, another output redefinition method called *Zero Magnitude Error Tracking Controller* (ZMETC) is introduced. It relies on the fact that $N_u(s)$ and $N_u(-s)$ have the same magnitude at all frequencies. The transfer function for such approximation is given by:

$$\hat{H}_3(s) \quad = \quad \frac{N_a(s)N_u(-s)}{D(s)} := C_a(sI - A_a)^{-1}B_a \tag{15}$$

where

$$\begin{aligned} A_a &:= A \\ B_a &:= B \\ C_a &:= \hat{C}_2 \\ x_a &:= x \end{aligned} \tag{16}$$

This approximate model will have zero magnitude error at all frequencies. On the other hand, it will exhibit phase error at all frequencies other than dc. The relative degree of this approximate model is $r_3 = r$.

FDZPETC Output Redefinition

If the desired output trajectory is a sinusoidal signal with zero dc component, then the ZPETC method can be modified to achieve perfect tracking control with internal stability and hence the name *Frequency Dependent ZPETC* (FDZPETC). Suppose the frequency of the desired output is w_d . The controller is designed to cancel both frequency and magnitude errors with internal stability. The redefined equation is given as:

$$\hat{C}_{1_{new}} adj(SI - A)^{-1} B = \frac{N_a(s)N_u(jw_d)N_u(-jw_d)}{N_u(0)} \tag{17}$$

$\hat{C}_{1_{new}}$ should replace $\hat{C}_1$ in the definition of A_a. Moreover, $\hat{C}_{1_{new}}$ can be easily calculated from $\hat{C}_1$ using:

$$\hat{C}_{1_{new}} = \frac{N_u(jw_d)N_u(-jw_d)}{N_u(0)^2}\hat{C}_1 \tag{18}$$

Control

If the desired output trajectory $y_d(t)$ is given, two possible controllers based on exact or approximate models can be designed.

Perfect Tracking Control

This controller, which is based on direct inversion of the original system, would result in perfect tracking control with internal stability only if the zeros of the transfer function are all acceptable. If at least one zero is unacceptable, then this kind of controller is not practical. The linear inverse control is:

$$
\begin{aligned}
u(t) &= \frac{v(t) - CA^{r}x(t)}{CA^{r-1}B} \\
v(t) &= -\sum_{j=0}^{r-1} K_j \left(y^{(j)}(t) - y_d^{(j)}(t)\right) + y_d^{(r)}(t)
\end{aligned}
\tag{19}
$$

or simply

$$
u(t) = \frac{-K_0\left(Cx(t) - y_d(t)\right) + \dot{y}_d(t) - CAx(t)}{CB} \tag{20}
$$

if the relative degree r is one, where $y_d^{(i)}(t)$ denotes the i^{th} derivative of y_d (t). It should be mentioned that the above controller assumes that the required output derivatives are calculated from the states using the mathematical model of the system.

Approximate Tracking Control

Even though approximate by construction, this controller will provide stable internal dynamics as well as good approximate output tracking. The linear control based on output redefinition method is:

$$
\begin{aligned}
u(t) &= \frac{v_a(t) - C_a A_a^{r_a} x_a(t)}{C_a A_a^{r_a - 1} B_a} \\
v_a(t) &= -\sum_{j=0}^{r_a - 1} K_j \left(\hat{y}^{(j)}(t) - y_d^{(j)}(t)\right) + y_d^{(r_a)}(t)
\end{aligned}
\tag{21}
$$

or simply

$$
u(t) = \frac{-K_0\left(C_a x_a(t) - y_d(t)\right) - K_1\left(C_a A_a x_a(t) - \dot{y}_d(t)\right) + y_d^{(2)}(t) - C_a A_a^{2} x_a(t)}{C_a A_a B_a} \tag{22}
$$

if the relative degree r_a is two.

This controller can be used for all of the above methods discussed in Output Redefinition Section with appropriate definitions of A_a, B_a, C_a, x_a, and r_a.

Simulation Example

As an example, two sates of the boost converter are defined by:

$$
\begin{aligned}
\dot{x}(t) &= \begin{cases} A_1 x(t) + B_1 u(t), & kT \le t < kT + d_k T \\ A_2 x(t) + B_2 u(t), & otherwise \end{cases} \\
y(t) &= Cx(t)
\end{aligned} \tag{23}
$$

where $x \in R^n$ is the state vector, $y \in R$ is a scalar output, and $U \in R$ is a scalar input. As indicated, the circuit dynamics switch between two topologies, (A_1, B_1) and (A_2, B_2), with switching period T and duty ratio $d_k \in [0,1]$, where k represents the discrete-time index. The constant matrices of the system are:

$$
A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \quad B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}
$$
$$
A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \quad B_2 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{24}
$$

The circuit component values are $R = 10\Omega$, $L = 2$ mH, $C = 200\mu$F and $U = 100$ V. If we operate close enough to the equilibrium point corresponding to $\bar{d}$ and assume infinite switching frequency, then the small signal model is a good approximation of the system and can be used to design the controller (a continuous duty ratio $d(t)$) (see [4]). The small signal model will also be considered as the exact model for the system as to apply output redefinition on a linear system. The small signal model at $\bar{d} = 0.5$ can be found to be:

$$
A = \begin{bmatrix} 0 & -250 \\ 2500 & -500 \end{bmatrix} \quad B = \begin{bmatrix} 1\times10^5 \\ -2\times10^5 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{25}
$$

The transfer function of this linear system is:

$$H(s) = \frac{-2\times10^5(s-1250)}{s^2+500s+625000} := \frac{N_a(s)N_u(s)}{D(s)} \tag{26}$$

where

$$\begin{aligned} N_a(s) &= -2\times10^5 \\ N_u(s) &= s-1250 \\ D(s) &= s^2+500s+625000 \end{aligned} \tag{27}$$

The relative degree is one and the number of unacceptable zeros is also one. Using the output redefinition method, the vectors $\hat{C}_1$ and $\hat{C}_2$ are easily calculated to be:

$$\hat{C}_1 = \frac{1}{9}\begin{bmatrix}10 & 5\end{bmatrix} \qquad \hat{C}_2 = \frac{1}{9}\begin{bmatrix}20 & 1\end{bmatrix} \tag{28}$$

The two types of controllers were used to track the desired output trajectory $y_d(t) = 200 - 30\sin(200\pi t)$. The simulation results for perfect tracking control are shown in Fig. 1 and the results of using approximate output tracking controls are shown for three versions of output redefinition approximation in Figs. 2, 3 and 4.

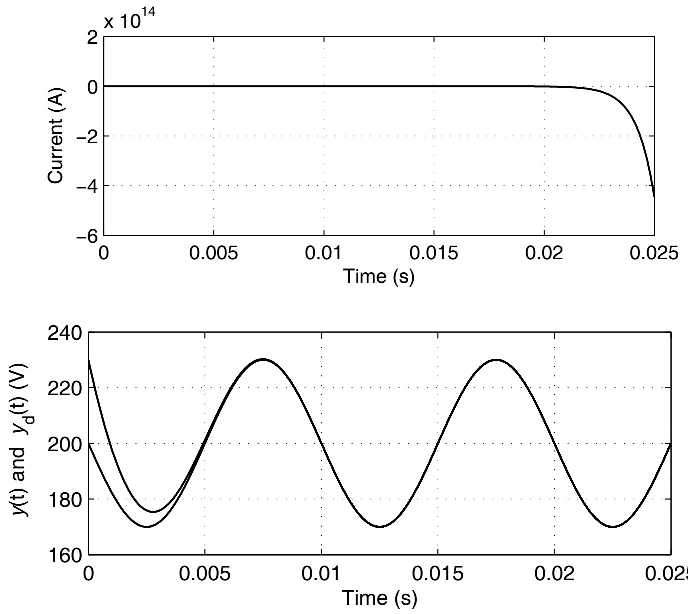

Fig. 1. Perfect output tracking for boost converter.

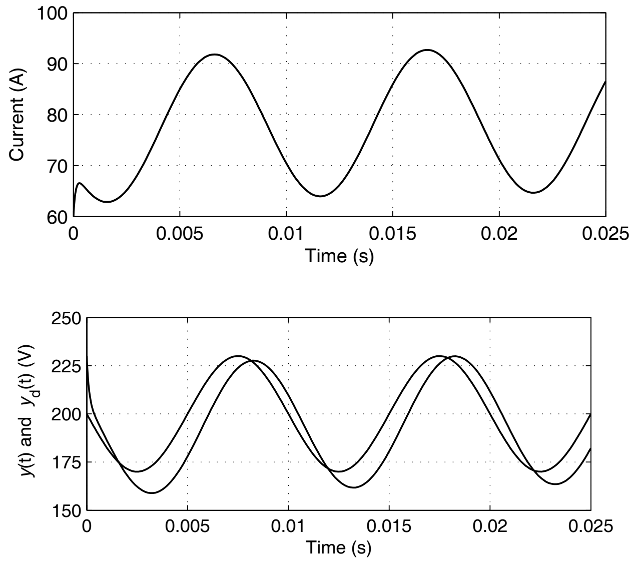

Fig. 2. ZDCETC for boost converter.

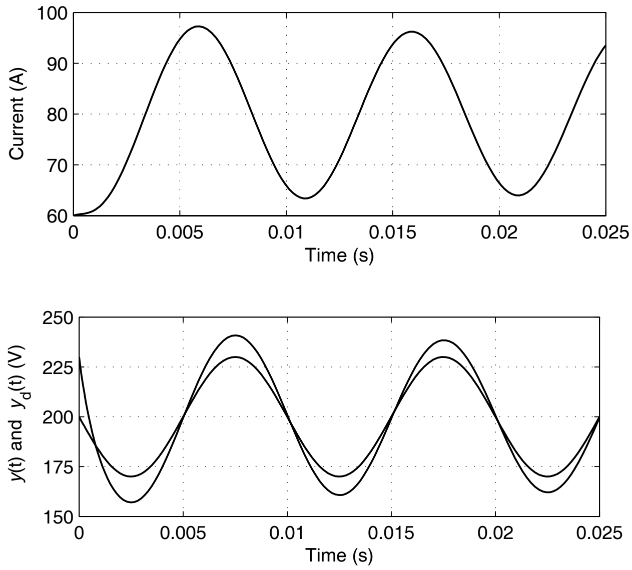

Fig. 3. ZPETC for boost converter.

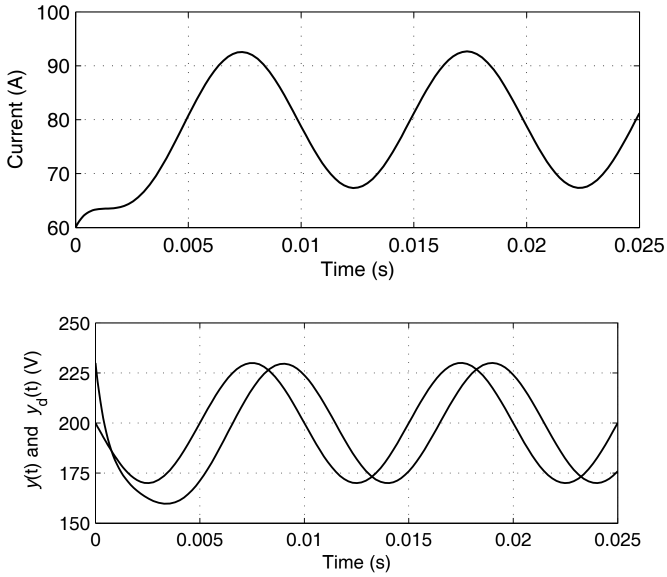

Fig. 4. ZMETC for boost converter.

If the desired output trajectory was $y_d(t) = -30\sin(200\pi t)$ and the controller was designed to cancel the tracking error at the frequency of $y_d(t)$, then $\hat{C}_{1_{new}}$ becomes:

$$\hat{C}_{1_{new}} = \frac{(200\pi)^2 + 1250^2}{1250^2} \frac{1}{9}\begin{bmatrix} 10 & 5 \end{bmatrix} \tag{29}$$

and the simulation results are shown in Fig. 5.

Conclusion

The modeling and control of linear systems with unacceptable zeros is problematic since it causes the internal dynamics of the system to blow up or to behave in an undesired manner. To solve this problem, three or four different methods based on output redefinition approximation are presented and shown to outperform the perfect tracking controller in the presence of unacceptable zeros. Simulation results show that the idea of output redefinition, although approximate, ensures the boundedness of internal dynamics. The knowledge of the desired output frequency allows the use of the FDZPETC method to cancel the error between the desired and actual output signals provided that the desired trajectory is a sinusoidal function with zero DC value. A further investigation of the effect of model uncertainty and output feedback instead of state feedback would be a good extension to this work.

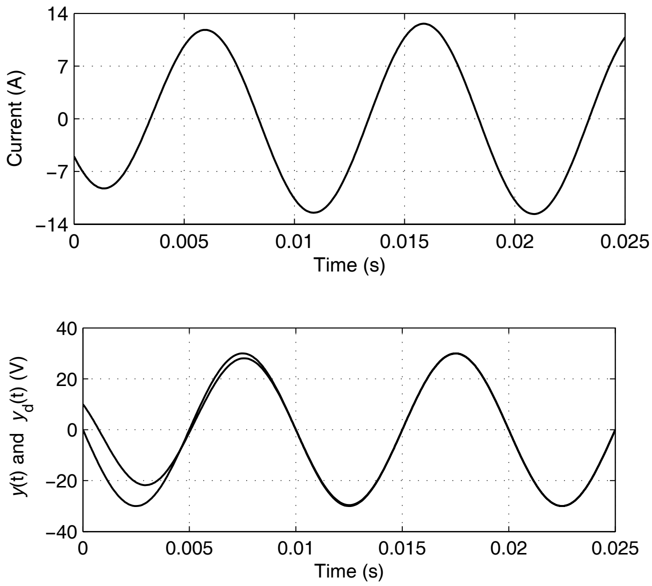

Fig. 5. ZMETC for boost converter.

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كم تتابعي تقري موحد للأنظمة ا طية وجود أصفار غ مقبولة

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ملخص البحث. تدرس الورقة مشكلة النمذجة والتحكم للأنظمة الخطية المتصلة بوجود أصفار غير مقبولة تكون في النصف الأيمن من نطاق لابلاس (طور غير أدنى) أو تتسبب في حدوث سلوك غير مرغوب فيه للنظام. سيؤدي التقريب الموحد للتحكم التتابعي لدالة الخرج باستخدام إعادة تعريف الخرج إلى تحكم تتابعي تقريبي، ولكنه في الوقت نفسه يضمن الاستقرار الداخلي للنظام. تم عرض أربع طرق مختلفة لإعادة تعريف الخرج بطريقة موحدة تؤدي كل واحدة منها إلى تحقيق هدف مختلف من أهداف التحكم. وفي نهاية الورقة تم تطبيق الطرق الأربع والتأكد من صحة الخلفية النظرية عن طريق مثال نمذجة للتقريب الخطي لمحول بوست.