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Modification of the Error Equation in Angular Theodolite Measurements

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Abstract. In this information era, automation of surveying operations becomes a must, simply because an excessive amount of information cannot always be handled manually. However, prior to automation, rigorous mathematical models should be formulated in such a way that all effective parameters are taken into consideration. This step is important, otherwise automation will always be defective. From this perspective, the authors examined the famous equation of the theoretical accuracy of theodolite angular measurement and reevaluated it by considering the effects of target type and observation distance length on the measuring process, beside that of the reading error, the pointing error and the initial setting error being taken care of in the original equation. A new mathematical model for estimating the theoretical angular error in theodolite works was then developed and tested.

Introduction

The concept of the design of an angle measuring and reading system seems to have been often overlooked as many surveying and civil engineers tend to base it on experience (i.e. measuring angles as they have always been done according to techniques, specifications and practices of governmental mapping agencies) rather than on specific statistical methods that take into account precision and/or accuracy of the survey work. For example, it is generally believed among surveyors that eight sets of backsight, and foresight of direction measurement with a $1''$ theodolite (e.g. Wild T2) or six sets with a $0.1''$ theodolite (e.g. Wild T3) should be made [1].

Of course an alternative to this undertaking is to design angular measurement systems prior to field measurements using appropriate information on individual system component performance and expected errors that may affect the whole measuring system. The various components involved are (i) the observer and his sight (ii) the optics of the theodolite (iii) the atmosphere and (iv) the target used and its characteristics (i.e. color, background, geometric shape, contrast, size, dimensions, etc.).

Therefore, in the design of angular measurement systems, if we know (or can logically assume) some of these errors, we can combine them by means of error propagation theories; and come up with reasonable values for them.

Theodolite Measurement Error Equation

If the instrument used for angular measurement has been adjusted for errors such as collimation, plate level, diaphragm misorientation, trunnin axis displacement and optical plummet, and if it has been carefully centered over the ground point and carefully levelled, then it is believed that the two sources of error remaining in horizontal angle measurement will be pointing the theodolite to the target and reading circle graduations [2].

An error equation describing this was first developed by Colcord [3] using the mathematical models of the measuring process and error propagation techniques. For n observations of an angle; the equation is as follows:

$$\sigma_c = \pm \frac{1}{\sqrt{n}} \sqrt{e_0^2 + 2e_p^2 + e_r^2} \qquad (1)$$

where

σ_c = the theoretical combined random error;
n = number of repetitions of measuring an angle;
e_o = error in initial setting of a theodolite on the target;
e_r = reading error (optical theodolites only); and
e_p = error made in pointing to the target .

Reading errors (e_r) are generally considered equal to ½ the smallest circle graduation [3]. Errors in initial setting are assumed equal to pointing errors. Also, experience shows that with well-adjusted equipment, an average observer can point the theodolite within 2" to 5" (i.e. e_p = 2" to 5"). This equation is now widely accepted as being representative of the amount of error that is to be expected in the design of angulation systems (see [2, 4, 5]).

Verification of the Theodolite Error Equation

Initially, it is appropriate to verify the validity of equation (1) by first carrying out a number of angular measurements with theodolites having different designs and capabilities. Also, targets of various types were used to attest the effect of target type on the measuring system. This is relevant because it seems that as long as the surveyor has to align theodolite cross hairs on physical targets, target selection must be included in the design of any angle measuring system. It also seems that as far as the survey targets are well-designed, well-lit, clear and stable over the survey point, both accuracy and precision should result. Assuming there is no differential horizontal refraction, such a test was carried out by the present two authors [6]. A 500m long, 5-section base-line was first established and marked by wooden stakes at 100m intervals. Horizontal angles at the one end of the line from a permanently-fixed nearby triangulation point to each of the wooden stakes were first established using a Wild T3 precision theodolite in conjunction with a Wild GZT-1 survey target. The angles were then remeasured ten times using various theodolites and targets and then compared with their T3 counterparts through computations of root-mean-square error (r.m.s.e) values [6]. Parts of these results are shown here as Tables 1, 2, 3 and 4 for a Wild T1, a Wild T2, a Kern DMK-1 and a Sokkia SET2C (total station) instruments, respectively. The first three instruments are of optical design reading to $6''$, $1''$, and $10''$,respectively, while the fourth is an electronic total station being used as an electronic theodolite reading to $1''$ using the angle measuring component of the instrument. The tables also show the types of targets used in the experiment. Further, using Eq.(1), Table 5 was compiled to represent the theoretical values expected to be achieved by the various instrument-target-distance combinations taking into account the assumed values for $e_{o,}$ $e_{r,}$ and e_p. Due comments on these Tables are as follows:

(i) For the Wild T1 theodolite, the expected theoretical error of angular measurement is $\sigma_c = \pm 1.9''$ (Table 5). However, Table 1 shows that this value has been achieved only with the cases of the geodetic target and the plumb bob string both being sighted at only 100 m distance from instrument station. The rest of the results on Table 1 deviate markedly from this figure.

(ii) For the Wild T2 theodolite observation, the corresponding expected theoretical error is approximately $\sigma_c = \pm 1.4''$ (Table 5). Table 2, on the other hand, shows that only the geodetic target and the plumb bob string (again observed at only 100 m distance) were able to achieve this value.

(iii) With the Kern DKM-1 theodolite, the theoretical error amounts to $\pm 2.6''$. Table 3, however, reveals the fact that none of the test targets achieved this accuracy. This is true for all observation distances.

(iv) The corresponding theoretical error for the SET2C total station is also approximately $\pm 1.4''$ (Table 5). It is, however, quite astounding to notice that none of the test targets were able to achieve this figure at any distance (see Table 4).

Table 1. Angular measurement accuracy (σ_c) with the Wild T1 (sec. of arc) [6]

Distance (m)	Geodetic target	Steel nail	Plumb bob string	Supported ranging rod	Hand-held ranging rod
100	±1.5	±2.2	±1.7	±3.4	±4.4
200	±2.5	±3.4	±3.7	±5.0	±5.3
300	±3.3	±3.7	±4.0	± 5.6	±6.0
400	±4.2	± 6.5	±6.3	±7.9	±8.5
500	±5.5	±8.3	±8.8	±11.7	±12.0

Table 2. Results (σ_c) obtained with the Wild T2 Theodolite (sec. of arc) [6]

Distance (m)	Geodetic target	Steel nail	Plumb bob String	Supported ranging rod	Hand-held ranging rod
100	±1.1	±1.5	±1.2	±1.8	±2.1
200	±1.5	±1.7	±1.8	±3.0	±3.7
300	±2.2	±2.9	±3.4	±4.7	±5.5
400	±2.9	±3.5	±3.7	±4.8	±5.6
500	±3.6	±5.0	±5.3	±5.8	±6.4

Table 3. Results obtained with the Kern DKM-1 Theodolite (sec. of arc) [6]

Distance (m)	Geodetic target	Steel nail	Plumb bob string	Supported ranging rod	Hand-held ranging rod
100	±3.3	±3.6	±4.5	±5.6	±5.8
200	±3.9	±4.3	±4.7	±6.0	±6.3
300	±6.5	±8.3	±7.9	±8.1	±8.4
400	±9.2	±12.5	±13.2	±12.2	±12.8
500	±15.8	NA	NA	±17.9	±18.8

NA=not available

Table 4. Results obtained with the Sokkia SET 2C total station (sec. of arc) [6]

Distance (m)	Geodetic target	Steel nail	Plumb bob string	Supported ranging rod	Hand-held ranging rod
100	±2.1	±2.3	±2.4	±2.7	±2.8
200	±2.3	±2.7	±2.9	±3.6	±3.9
300	±2.2	±2.5	±2.8	±3.6	±4.0
400	±2.9	±3.8	±3.7	±4.6	±5.1
500	±4.0	±4.4	±5.1	±5.6	±6.2

Table 5. Expected theoretical errors of angular measurement (n=10)

Instrument	e_o, e_r (seconds)	Combined theoretical error (seconds)
Wild T3	0.05	1.34
Wild T1	3	1.9
Wild T2	0.5	1.36
Kern DKM-1	5	2.6
Sokkia SET2C	0.5	1.35

Taking these findings into consideration and noting that Eq. (1) does not in fact take into account type of target used and/or distance of observation, (although it does cater for theodolite type and observer capability), one would suggest modification of Eq.(1) to accommodate the effects of these two parameters.

Modification of the Error Equation

To account for type of target used in angular measurements, an experiment was conducted indoors (i.e. in a stable, and constant-temperature environment) in which a small angle was first established twenty times using a recently-checked Wild T3 precision theodolite in conjunction with two Wild GZT-1 geodetic targets on the two ends of the legs of the angle. In order to minimize bisection errors caused by targets being overmagnified (see [7, 8]), the two target stations were chosen to be more than 6 m from theodolite station. The choice of a small angle was based on the fact that measuring larger angles may increase the effect of circle graduation errors for optical reading theodolites.

The precision of establishing this angle (computed as standard deviation from the mean) was in the order of $\pm 0.4''$ and was viewed as satisfactory for the purpose of the test (The German DIN 18723 specifications state ± 0.5"). This precision value was considered as reference and is denoted here as σ_f.

Leaving the left-hand target in its position, the right-hand target was then replaced by each one of the test targets shown on Tables 1, 2, 3 and 4 in turn and the angle was then remeasured twenty times as before using the T3 theodolite. The ranging rod was used in two distinct configurations. In the first set-up of this target, the rod is firmly secured and supported on a tripod. In the second case (i.e. the case of the unsupported rod), the rod is hand-held. The rod-man endeavored to hold this target erect and vertical by attaching a " bond level" to it while it is being sighted by the instrument man. A surveyor with more than twenty years experience carried out the observations. This is believed to be an advantage since it tends to minimize human errors in angular measurement.

The precision of angular measurement using a particular target was then computed as standard deviation from the mean (σ_i); and the ratio σ_i/σ_f was then adopted as a constant representing the effect of target-type on the measuring system i.e.

$$k_i = \frac{\sigma_i}{\sigma_f} \tag{2}$$

where

k_i = constant representing target type i;

σ_i = precision of angular measurement using target i (i=steel nail, plumb bob string, supported rod and unsupported rod respectively).

Table 6 shows the values of this constant for the various targets used in this test.

Table 6. values of target constant k

Type of target	k
Geodetic	1.000
Steel nail	1.032
Plumb bob	1.040
Supported rod	1.210
Unsupported rod	1.530

This factor (k) was then applied to Eq. (1) as a multiplicative constant.

To account for the effect of observation distance on the measuring process, it was assumed that the angular error is proportional to the cubic root of the distance to the target, i.e.

$$\sigma_c \propto \sqrt[3]{d} \tag{3}$$

Where d = distance of observation in meters.

For convenience, Eq.(3) is rearranged so that

$$\sigma_c \propto \sqrt{d^{2/3}} \tag{4}$$

Combining Eqs. (1), (2) and (4), a new equation for the theoretical error in angular measurement can be developed as follows:

$$\sigma_c = \pm k\sqrt{\frac{d^{2/3}}{n}(e_o^2 + 2e_p^2 + e_r^2)} \tag{5}$$

where

σ_c = the new expected theoretical error in measuring an angle with a theodolite; and k, n, d, e_o, e_p, e_r are as before.

Verification of the Modified Error Equation

Tables 8, 9 and 10 have been compiled using the modified error Eq.(5), and represent the ranges of the maximum expected combined theoretical random error in angular measurements carried out with various theodolite/target/distance combinations. The tables are clearly self-explanatory. However, it seems appropriate to augment them with additional remarks.

(i) With observations being carried out using the Wild T1 (Table 7), the expected theoretical accuracy at 100m distance ranges from ±8.8" with the geodetic target to ±13.5" with the hand-held ranging rod. The corresponding values at 200m distance are ±11.1" and ±17.0"; and ±15.1"and ±23.0" at the terminal end of the line (500m). Comparing the contents of Table 7 with those presented on Table 1 for the Wild T1 theodolite, it is clear that all test targets are now able to achieve results within the expected theoretical accuracy.

(ii) When the Wild T2 theodolite is used to sight the test targets at 100 m distance, the resulting theoretical random error in angular measurement was obtained as follows (Table 8):

$\sigma_c = \pm 6.3''$ for the geodetic target;
$\sigma_c = \pm 6.5''$ for the steel nail;
$\sigma_c = \pm 6.6''$ for the plumb bob string;
$\sigma_c = \pm 7.7''$ for the ranging rod being supported on tripod, and
$\sigma_c = \pm 9.7''$ with the hand-held ranging rod.

Comparing these values with those shown on Table 2, it is clear that all test targets were able to achieve results well within the expected range of error. The same argument applies to other sighting distances used in the present test. However, it is noted that the theoretical errors expected with the Wild T1 being used for angular observations are, in general, larger than those expected when Wild T2 is used. This is logical since T2 is a double-reading micrometer instrument with only 1" least count while T1 is a single-reading micrometer theodolite with a least count value of 6" (the corresponding DIN 18723 specifications are ±0.8" and ±3" respectively).

(ii) It is noted that for all test targets the expected theoretical accuracy deteriorates gradually and smoothly as observation distance is increased. This is in line with common experience.

(iv) With the Kern DKM-1 theodolite (Table 9), the expected theoretical accuracy values range from $\sigma_c = \pm 12.1''$ with the geodetic target observed at 100m distance to $\sigma_c = \pm 21.7''$ with the hand-held ranging rod being sighted at 500m distance. Here, the expected angular errors are, generally speaking, higher than those expected with the Wild T1. An obvious explanation is that the Kern DKM-1, although also a single-reading micrometer theodolite, has a least count value of 10'' (corresponding to DIN 18723 specification of ±5" instead of 3" for the T1). Nevertheless contrasting the contents of Tables 3 and 9, it is evident that at all sighting distances of the test, all test targets were able to give accuracy figures well within the maximum expected range.

(v) For the Sokkia SET2C total station, the reading error component of Eq. (5) is of course equal to zero since circle reading is carried out electronically. Table 10 shows the expected theoretical errors using this instrument. Thus values of $\pm 6.2''$, $\pm 6.5''$, $\pm 6.5''$, $\pm 7.8''$, and $\pm 9.6''$ are expected for the geodetic target, the steel nail, the supported rod and the hand-held rod observed at 100 m distance. A glance at Table 4 shows that the test targets did actually achieve results within the expected random error. This same argument is true at the 200m, the 300m, the 400m, and the 500m observation distances.

Conclusion

The paper addressed the question of the theoretical combined error in angular measurements using a theodolite. A formula for this parameter was originally developed by Colcord [3] (Eq.(1)) using the theory of propagation of errors. In an earlier experiment concerned with evaluation of the accuracy of angular measurement with various targets at different distances, the authors noted that only in few cases (i.e. namely the cases of the geodetic target, the steel nail and the plumb bob observed at 100m and sometimes 200 m using T2 or T1 theodolites) were the various theodolite/target/ distance combinations able to achieve accuracy figures computed by Colcord [3] and Kissam [1] based on Colcord's formula. It was then thought to modify Colcord's equation by making it take into account the effects of target type and distance of observation on the measuring process.

For this purpose an experiment was conducted indoors in order to compute constants representing the effects of target-type on angular measurement. The error contributed by variation in observation distance was assumed to be proportional to the cubic root of observation distance. These were then combined with the original formula developed by Colcord to obtain a new formula for the theoretical random error. The theoretical accuracies of the various combinations were then computed using the generally accepted values for reading, pointing and initialization errors. In all cases considered, it was found that all test targets were able to achieve accuracy values well within the expected theoretical range as developed in this study. The authors therefore appeal to surveyors to use this new formula in their respective works when attempting angulation system designs.

Table 7. Expected theoretical accuracy with the Wild T1 theodolite

Distance	Geodetic target k=1.00	Steel nail k=1.032	Plumb bob string k=1.04	Supported ranging rod k =1.21	Hand-held rod k=1.53
100	±8.8"	± 9.1"	±9.2"	±10.7"	±13.5"
200	±11.1"	±11.5"	±11.5"	±13.4"	±17.0"
300	±12.7"	±13.1"	±13.2"	±15.4"	±19.4"
400	±13.9"	±14.4 "	±14.5"	±16.9"	±21.4"
500	± 15.1"	±15.5"	±15.7"	±18.2"	±23.0"

Table 8. Expected theoretical error with the Wild T2 observations

Distance	Geodetic target k=1.00	Steel nail k=1.032	Plumb bob string k=1.04	Supported ranging rod k =1.21	Hand-held rod k=1.53
100	± 6.3 "	± 6.5"	±6.6"	±7.7"	±9.7"
200	± 8.0"	±8.2"	±8.2"	± 9.7"	±12.2"

300	± 9.1″	±9.4″	±9.5″	±11.2″	±14.0″
400	± 10.1″	±10.3″	±10.4″	±12.2″	±15.4″
500	±10.8″	±11.1″	±11.2″	±13.2″	±16.6″

Table 9. Combined random error with the Kern DKM-1 theodolite

Distance	Geodetic target k=1.00	Steel nail k=1.032	Plumb bob string k=1.04	Supported ranging rod k =1.21	Hand-held rod k=1.53
100	±12.1″	±12.5″	±12.6″	±14.7″	±18.5″
200	±15.3″	±15.8″	±15.9″	±18.5″	±23.3″
300	±17.5″	±18.0″	±18.2″	±21.1″	±26.7″
400	±19.2″	±19.8″	±20.0″	±23.3″	±29.4″
500	±21.7″	±21.4″	±21.5″	±25.1″	±31.7″

Table 10. Expected theoretical error with the Sokkia SET2C

Distance	Geodetic target k=1.00	Steel nail k=1.032	Plumb bob string k=1.04	Supported ranging rod k =1.21	Hand-held rod k=1.53
100	± 6.2″	±6.5″	±6.5″	±7.8″	±9.6″
200	±7.9″	±8.2″	±8.2″	±9.6″	±12.1″
300	±9.0″	±9.3″	±9.4″	±10.9″	±13.8″
400	±9.9″	±10.2″	±10.4″	±12.0″	±15.8″
500	±10.6″	±11.1″	±11.2″	±13.0″	±16.4″

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تطوير معادلة الخطأ في قياس الزوايا بالثيودوليت

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ملخص البحث. في عصر المعلومات أصبحت الحاجة ملحّة لأخذ المعالجة الآلية المتكاملة في العمل المساحي بعين الاعتبار لأنه لم يعد من السهل التعامل يدويًا أو جزئيًا مع المعلومات المساحية هائلة الكم كما كان الحال في السابق مع المعلومات قليلة الكم. ولا شك أن من متطلبات هذا النهج الملح تصميم وتطوير نماذج رياضية عالية الدقة والكفاءة وتطويرها آخذة في الاعتبار ما أمكن من العوامل المؤثرة في هذه المعلومات. ذلك لأن عدم وجود هذه النَّماذج يقود حتمًا إلى عدم جدوى العمليات المساحية الآلية برمتها. في هذا المنحى قام المؤلفان بتقويم معادلة الخطأ المشهورة لقياس الزوايا بالثيودوليت وذلك باختبارها باستعمال عدد من الأهداف المساحية على مسافات طويلة مختلفة دون إغفال العوامل المعتبرة في المعادلة الأصلية التي هي خطأ القراءة، وخطأ توجيه الجهاز، وخطأ التصفير في الدائرة الأفقية إضافة إلى عاملي بعد المسافة ونوعية الهدف. وعلى هذا الأساس قام المؤلفان باشتقاق معادلة جديدة مطوَّرة لخطأ القياس بالثيودوليت. اختبرت هذه المعادلة نظريًا وعمليًا فأثبتت التَّجربة أنها المعادلة الأصح لحساب خطأ القياس.