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# Variable-entered Karnaugh Map Procedures for Obtaining the Irredundant Disjunctive Forms of a Switching Function from Its Complete Sum

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**Abstract.** The covering problem, a classical problem in switching theory, involves the selection of some prime implicants irredundantly to cover the asserted part of an incompletely specified switching function. This paper advocates the use of the variable-entered Karnaugh map (*VEKM*) to solve this problem. To set the stage for achieving this purpose, a novel tutorial exposition of the *VEKM*-related variable-entered cover matrix is presented. Subsequently, the paper offers two *VEKM* solutions of the covering problem. The first solution relies on the new concept of *VEKM* loops. By contrast to a loop on a conventional Karnaugh map which is characterized by a single product, a *VEKM* loop is identified by two products, namely loop coverage (which depends only on map variables) and loop depth (which depends only on entered variables). In the second solution, the VEKM is used directly as a cover map in such a way that it is not littered with unnecessary details or too many overlapping loops. All solution procedures are demonstrated via illustrative examples that help reveal their visual merits as well as their applicability to relatively large problems.

## 1. Introduction

The covering problem is a classical problem in switching theory that involves the determination of all irredundant disjunctive forms (*IDFs*) of a switching function from its complete sum. Though interest in this problem for digital-design purposes has declined, the covering problem still remains of paramount importance in many other engineering applications such as the areas of Boolean reasoning and Boolean-equation solving [1,2]. In these areas, the analyst typically handles medium-sized problems and may occasionally prefer an insightful manual tool to a faster software one.

We deal in this paper with an incompletely specified switching function expressed in the form $f = g \vee d(h)$, where $g$ and $h$ are completely specified switching functions, called the asserted and don't care parts of $f$, respectively. An irredundant disjunctive form (*IDF* ) of $f$ is a minimal subformula of the complete sum *CS* for $(g \vee h)$ that covers $g$ [1]. In other words, a sum-of-products (s-o-p) formula $S$ is an *IDF* for $f$ if and only if

(a) $S$ is a subformula of $CS(g \vee h)$, i.e., each and every term of $S$ is a term of $CS(g \vee h)$ and hence a prime implicant of $f$.

(b) $g \leq S$, i.e., the formula $S$ covers the asserted part of $f$.

(c) No proper subformula of $S$ has the property (b), i.e., if one or more terms of the formula $S$ is removed from it, it ceases to cover the asserted part of $f$.

Among the *IDFs* of $f$, any formula with the minimum number of terms as a primary criterion, and with the minimum number of literals as a secondary criterion is called a minimal sum for $f$. While $f$ has a unique complete sum it may have more than one minimal sum.

There are many algebraic, tabular or mapping methods for obtaining all the *IDFs* of a switching function [3]. Most of these methods use the complete sum as a starting point, or more generally act in a 2-step fashion by finding the complete sum first before proceeding to derive the *IDFs*. To the list of these methods, it is desirable to add one method based on the use of the variable-entered Karnaugh map (*VEKM*) since such a method could be (a) an efficient combination of algebraic and mapping methods, (b) a powerful tool that can push further the limit on the number of variables to be manually processed, and (c) a pedagogical aid that provides pictorial insight on the intrinsic structure of the problem and on the intricacies of its proposed solutions. In a companion paper [4], the *VEKM* has been used successfully to address the first step of deriving the complete sum. In this paper, the work of [4] is continued by exploring use of the *VEKM* in obtaining all the *IDFs* of an *ISSF* $g \vee d(h)$ from its complete sum $CS(g \vee h)$. Once the set of all *IDFs* is obtained, the minimal sum is trivially deduced. To set the stage for this work, we start in section II by presenting a novel tutorial exposition of the *VEKM*-related Variable-Entered Cover Matrix (*VECM*) introduced in [5]. This *VECM* exposition is made in such a way that its ideas can be lucidly carried out to *VEKM* procedures. Two such procedures are given herein in detail, one in section III that introduces a new concept of *VEKM* loops and discusses the construction of *VEKM* prime-implicant loops, and the second in section IV that uses the *VEKM* as a cover map. The present *VEKM* procedures are algorithmic in nature and can be easily automated. They guarantee the derivation of all *IDFs* of an *ISSF* and consequently obtain its exactly minimal sum(s). By contrast, earlier single-step *VEKM* procedures (see, e.g. [6]) are heuristics that do not achieve exact minimality all the time. Final comments are given in section V.

## 2. A *VEKM*-related Exposition of the Variable-entered Cover Matrix (*VECM*)

Reusch [5] used a variable-entered cover-matrix to select sets of prime implicants such that their sums are *IDFs* of a given *ISSF*. His method provides a common setting in which to view and/or generalize the earlier results of Quine [7], McCluskey [8], Ghazala [9], Chang and Mott [10]. Moreover, Reusch's method does not restrict the nature of the representation of the switching function under consideration; neither does it require that such a representation be related in any fashion to the complete sum of the function. However, Reusch's paper is somewhat hard to follow, though it can be given a lucid clear interpretation that relies heavily on *VEKM*-related terms. It is such an interpretation that we hope to present to the reader now. The reader may find that our language is surprisingly different from that of Reusch, though we are delivering the essence of his method. Basically, he stressed the theoretical foundation of the method, while we are trying to give a practical working knowledge of it. In the two sections to follow, we will be able to develop the present method into what can be called purely *VEKM* cover methods.

To construct the Reusch cover matrix, let the asserted part of the *ISSF* $f$ under consideration be given by the s-o-p formula

$$g = \bigcup_{j}^{j_{max}} g_j, \tag{1}$$

where, beside being an asserted implicant of $f$, the term $g_j$ is not restricted in any other fashion whatsoever. In particular, $g_j$ is not required (though it can be) a minterm or a prime implicant. The complete sum for $(g \vee h)$ is given as

$$CS\,(g \vee h) = \bigcup_{i}^{i_{max}} P_i, \tag{2}$$

where the union operator in (2) runs over all the prime implicants $P_i$ of $(g \vee h)$. Now, construct a table or matrix of size $i_{max} \times j_{max}$ such that $P_i$ is the coordinate or key of the horizontal row $i$ of the table, while $g_j$ is the coordinate of its vertical column $j$. The entry $T_{ij}$ of the table at row $i$ and column $j$ is given by the ratio (residue or quotient)

$$T_{ij} = P_i/g_j = (P_i)_{g_j=1}. \tag{3}$$

Note that $T_{ij}$ represents the part of $g_j$ that is covered by $P_i$, and hence $\{T_{ij} = 1\}$ means that $P_i$ covers $g_j$ completely because $g_j$ subsumes $P_i$ while $\{T_{ij} = 0\}$ means that $P_i$ does not cover any part of $g_j$ because $P_i$ and $g_j$ are disjoint. For this latter case, Reusch uses a hyphen (—) instead of a 0, which does not conform to our present interpretation and seems somewhat awkward.

With the matrix $[T]$ properly constructed, we are sure that every column $j$ will have the value 1 in at least one of its elements, which is another way of saying that an asserted implicant $g_j$ of $f$ must be totally covered by at least one of its prime implicants $P_i$. If such a way of covering is the sole way for $g_j$ to be covered, then the corresponding $P_i$ is an essential prime implicant. However, it might so happen that $g_j$ is covered by some alternative means if several prime implicants $P_i$ 's cooperate to cover a single $g_j$, such that:

a) not a single one of them is capable of covering $g_j$ alone, and

b) if one of them is removed, then the remaining ones will not suffice to cover $g_j$.

For such a set of prime implicants both of the following two equivalent conditions are true:

a) the generalized consensus of these prime implicants is subsumed by $g_j$.

b) the union of table entries in column $j$ at the rows $i$ corresponding to these $P_i$ 's is equal to 1 irredundantly( The problem of finding minimal sets of terms that sum to one irredundantly is known as the tautology or the sum-to-one problem in logic theory [1] ).

The next step in the Variable-Entered Cover Matrix (*VECM*) method is to write a Petrick function *PF* expressing all the *IDFs* of *f*. This function is written as a product-of-sums formula:

$$PF = \bigcap_{j=1}^{j_{max}} A_j, \tag{4}$$

where $A_j$ is an expression of the possible ways of selecting prime implicants to cover the asserted implicant $g_j$, viz.,

$$A_j = \bigcup_{I} \bigcap_{i \in I} P_i, \tag{5}$$

with the union operator in (5) taken over all sets $I$ that satisfy the 2 properties

$$I \subseteq IT = \{1,2,3,\ldots, i_{max}\}, \tag{6a}$$

$$\bigcup_{i \in I} T_{ij} = 1 \;(\text{irr.}), \tag{6b}$$

where the abbreviation (irr.) stands for " irredundantly".

To find all *IDFs* of f, the Petrick function *PF* must be converted into a sum-of-products form by multiplying its alterms out and invoking well-known simplification rules (see, e.g., Muroga [3]). The details of the method are now illustrated by an example.

**Example 1**

Consider an *ISSF* whose asserted part is given by the unrestricted s-o-p formula

$$
\begin{aligned}
g = {} & \overline{X}_1\overline{X}_3X_4X_5 \vee X_1X_2\overline{X}_3X_4X_6 \vee X_1\overline{X}_2\overline{X}_3X_5X_6 \vee \\
& X_1\overline{X}_2X_3X_5\overline{X}_6 \vee X_1X_2X_3X_5X_6 \vee \overline{X}_1X_2X_3X_4X_5 \vee \overline{X}_1\overline{X}_2X_3X_7,
\end{aligned}
\tag{7}
$$

and whose upper bound $(g \vee h)$ is a *CSSF* which is characterized by a complete sum having the 16 prime implicants listed as row coordinates of Table 1. This Table acts as the *VECM* for this function. As expected, each column has one or more of its elements equal to 1. The Petrick function is immediately written as

$$
\begin{aligned}
PF &= (P_5 \vee P_1P_3 \vee P_6P_3)(P_1)(P_2 \vee P_3)(P_2)(P_7 \vee P_8)(P_6)(P_4) \\
&= P_1P_6(P_5 \vee P_3 \vee P_3)P_2\,P_4(P_7 \vee P_8) \\
&= P_1P_2\,P_4\,P_6\,(P_3 \vee P_5)(P_7 \vee P_8) \\
&= P_1P_2P_3P_4P_6P_7 \vee P_1P_2P_3P_4P_6P_8 \vee P_1P_2P_4P_5P_6P_7 \vee P_1P_2P_4P_5P_6P_8,
\end{aligned}
\tag{8}
$$

which means that the *ISSF* under consideration has 4 *IDFs*, all of which turn out to be minimal since all of them have the same number of terms and the same number of literals.

**Example 2**

Consider an *ISSF* whose asserted part is given by the disjunction of the first 8 prime implicants of the function in Example 1, and whose complete sum consists of the same 16 prime implicants in Table 1. The *VECM* of this function is given in Table 2. Now, the 1 entries appear exactly once per column. The Petrick function in this case is

$$
\begin{aligned}
PF &= (P_5 \vee P_1P_3 \vee P_6P_3)(P_1)(P_2 \vee P_3)(P_2)(P_7 \vee P_8)(P_6)(P_4) \\
&= P_1P_6(P_5 \vee P_3 \vee P_3)P_2\,P_4(P_7 \vee P_8) \\
&= P_1P_2\,P_4\,P_6\,(P_3 \vee P_5)(P_7 \vee P_8) \\
&= P_1P_2P_3P_4P_6P_7 \vee P_1P_2P_3P_4P_6P_8 \vee P_1P_2P_4P_5P_6P_7 \vee P_1P_2P_4P_5P_6P_8,
\end{aligned}
\tag{9}
$$

which means that the *ISSF* under consideration has a single *IDF* and hence a unique minimal sum.

**Example 3**

Consider the *ISSF* $f$ given in Example 1 of [6], namely

$$
f = \overline{B}C\overline{D} \vee A\overline{B}C \vee \overline{A}B\overline{C} \vee B\overline{C}D \vee d\left(\overline{A}\,\overline{B}\,\overline{C} \vee \overline{A}BC\overline{D} \vee ABCD\right).
\tag{10}
$$

**Table 1. The variable-entered cover matrix for a given *ISSF*, whose asserted part is in an unrestricted s-o-p form**

| | $\bar{X}_1\bar{X}_3X_4X_5$ | $X_1X_2\bar{X}_3X_4X_6$ | $X_1\bar{X}_2\bar{X}_3X_5X_6$ | $X_1\bar{X}_2X_3X_5\bar{X}_6$ | $X_1X_2X_3X_5X_6$ | $\bar{X}_1X_2X_3X_4X_5$ | $\bar{X}_1\bar{X}_2X_3X_7$ |
|---|---|---|---|---|---|---|---|
| $P_1 = X_2\bar{X}_3X_4$ | $X_2$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $P_2 = X_1\bar{X}_2X_5$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| $P_3 = \bar{X}_2\bar{X}_3X_5$ | $\bar{X}_2$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $P_4 = \bar{X}_1\bar{X}_2X_3X_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $P_5 = \bar{X}_3X_4X_5$ | 1 | $X_5$ | $X_4$ | 0 | 0 | 0 | 0 |
| $P_6 = \bar{X}_1X_2X_4X_5$ | $X_2$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $P_7 = X_1X_3X_5X_6$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $P_8 = X_2X_3X_5X_6$ | 0 | 0 | 0 | 0 | 1 | $X_6$ | 0 |
| $P_9 = \bar{X}_1X_2X_3\bar{X}_4X_6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_{10} = \bar{X}_2X_5X_7$ | $\bar{X}_2X_7$ | 0 | $X_7$ | $X_7$ | 0 | 0 | $X_5$ |
| $P_{11} = \bar{X}_1X_4X_5X_7$ | $X_7$ | 0 | 0 | 0 | 0 | $X_7$ | $X_4X_5$ |
| $P_{12} = \bar{X}_1X_3\bar{X}_4X_6X_7$ | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{X}_4X_6$ |
| $P_{13} = X_3X_5X_6X_7$ | 0 | 0 | 0 | 0 | $X_7$ | $X_6X_7$ | $X_5X_6$ |
| $P_{14} = X_1X_4X_5X_6$ | 0 | $X_5$ | $X_4$ | 0 | $X_4$ | 0 | 0 |
| $P_{15} = X_2X_4X_5X_6$ | $X_2X_6$ | $X_5$ | 0 | 0 | $X_4$ | $X_6$ | 0 |
| $P_{16} = X_4X_5X_6X_7$ | $X_6X_7$ | $X_5X_7$ | $X_4X_7$ | 0 | $X_4X_7$ | $X_6X_7$ | $X_4X_5X_6$ |

**Table 2. The variable-entered cover matrix for an *ISSF*, whose asserted part is the disjunction of prime implicants**

| | $X_2\bar{X}_3X_4$ | $X_1\bar{X}_2X_5$ | $\bar{X}_2\bar{X}_3X_5$ | $\bar{X}_1\bar{X}_2X_3X_7$ | $\bar{X}_3X_4X_5$ | $\bar{X}_1X_2X_4X_5$ | $X_1X_3X_5X_6$ | $X_2X_3X_5X_6$ |
|---|---|---|---|---|---|---|---|---|
| $P_1 = X_2\bar{X}_3X_4$ | 1 | 0 | 0 | 0 | $X_2$ | $\bar{X}_3$ | 0 | 0 |
| $P_2 = X_1\bar{X}_2X_5$ | 0 | 1 | $X_1$ | 0 | $X_1\bar{X}_2$ | 0 | $\bar{X}_2$ | 0 |
| $P_3 = \bar{X}_2\bar{X}_3X_5$ | 0 | $\bar{X}_3$ | 1 | 0 | $\bar{X}_2$ | 0 | 0 | 0 |
| $P_4 = \bar{X}_1\bar{X}_2X_3X_7$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $P_5 = \bar{X}_3X_4X_5$ | $X_5$ | $\bar{X}_3X_4$ | $X_4$ | 0 | 1 | $\bar{X}_3$ | 0 | 0 |
| $P_6 = \bar{X}_1X_2X_4X_5$ | $\bar{X}_1X_5$ | 0 | 0 | 0 | $\bar{X}_1X_2$ | 1 | 0 | $\bar{X}_1X_4$ |
| $P_7 = X_1X_3X_5X_6$ | 0 | $X_3X_6$ | 0 | 0 | 0 | 0 | 1 | $X_1$ |
| $P_8 = X_2X_3X_5X_6$ | 0 | 0 | 0 | 0 | 0 | $X_3X_6$ | $X_2$ | 1 |
| $P_9 = \bar{X}_1X_2X_3\bar{X}_4X_6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\bar{X}_1\bar{X}_4$ |
| $P_{10} = \bar{X}_2X_5X_7$ | 0 | $X_7$ | $X_7$ | $X_5$ | $\bar{X}_2X_7$ | 0 | $\bar{X}_2X_7$ | 0 |
| $P_{11} = \bar{X}_1X_4X_5X_7$ | $\bar{X}_1X_5X_7$ | 0 | $\bar{X}_1X_4X_7$ | $X_4X_5$ | $\bar{X}_1X_7$ | $X_7$ | 0 | $\bar{X}_1X_4X_7$ |
| $P_{12} = \bar{X}_1X_3\bar{X}_4X_6X_7$ | 0 | 0 | 0 | $\bar{X}_1\bar{X}_4X_6$ | 0 | 0 | 0 | $\bar{X}_1\bar{X}_4X_7$ |
| $P_{13} = X_3X_5X_6X_7$ | 0 | $X_3X_6X_7$ | 0 | $X_5X_6$ | 0 | $X_3X_6X_7$ | $X_7$ | $X_7$ |
| $P_{14} = X_1X_4X_5X_6$ | $X_1X_5X_6$ | $X_4X_6$ | $X_1X_4X_6$ | 0 | $X_1X_6$ | 0 | $X_4$ | $X_1X_4$ |
| $P_{15} = X_2X_4X_5X_6$ | $X_5X_6$ | 0 | 0 | 0 | $X_2X_6$ | $X_6$ | $X_2X_4$ | $X_4$ |
| $P_{16} = X_4X_5X_6X_7$ | $X_5X_6X_7$ | $X_4X_6X_7$ | $X_4X_6X_7$ | $X_4X_5X_6$ | $X_6X_7$ | $X_6X_7$ | $X_4X_7$ | $X_4X_7$ |

Now, let the asserted part of $f$ be expanded as a disjunction of minterms. The *VECM* of this function is given in Table 3 in which the column coordinates are the 6 asserted minterms of $f$, and the row coordinates are its 7 prime implicants which are obtainable from (10) by any of the techniques in [3]. Table 3 is a special case of a *VECM* which is constant rather than variable entered and is well known as the Quine-McCluskey cover matrix (*QMCM*) [3]. The *QMCM* is related to a *VECM* in the same way a *CKM* is related to a *VEKM*. The Petrick function in this case is

$$\begin{aligned} \mathrm{PF} &= (P_1 \vee P_4)(P_2 \vee P_4)(P_2 \vee P_7)(P_3 \vee P_7)(P_3 \vee P_6)(P_1 \vee P_5) \\ &= (P_1 \vee P_4 P_5)(P_2 \vee P_4 P_7)(P_3 \vee P_6 P_7) \\ &= P_1 P_2 P_3 \vee P_1 P_2 P_6 P_7 \vee P_1 P_3 P_4 P_7 \vee P_1 P_4 P_6 P_7 \vee P_2 P_3 P_4 P_5 \vee \\ &\quad P_3 P_4 P_5 P_7 \vee P_4 P_5 P_6 P_7, \end{aligned} \tag{11}$$

which means that the present *ISSF* has 7 *IDFs*, only one of which is minimal.

**Table 3. The variable-entered cover matrix for an *ISSF*, whose asserted part is the disjunction of minterms**

| | $\bar{A}B\bar{C}D$ | $\bar{A}B\bar{C}\bar{D}$ | $\bar{A}\bar{B}C\bar{D}$ | $A\bar{B}C\bar{D}$ | $A\bar{B}CD$ | $AB\bar{C}D$ |
|---|---|---|---|---|---|---|
| $P_1 = B\bar{C}D$ | 1 | 0 | 0 | 0 | 0 | 1 |
| $P_2 = \bar{A}\bar{D}$ | 0 | 1 | 1 | 0 | 0 | 0 |
| $P_3 = A\bar{B}C$ | 0 | 0 | 0 | 1 | 1 | 0 |
| $P_4 = \bar{A}\bar{C}$ | 1 | 1 | 0 | 0 | 0 | 0 |
| $P_5 = ABD$ | 0 | 0 | 0 | 0 | 0 | 1 |
| $P_6 = ACD$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $P_7 = \bar{B}C\bar{D}$ | 0 | 0 | 1 | 1 | 0 | 0 |

## 3. *VEKM* Prime Implicant Loops

This section discusses a new *VEKM* method for obtaining all the *IDFs* of an *ISSF* with a preknown complete sum. The basic idea is to draw a *VEKM* representation of the asserted part of the given function and then draw entered loops on the given *VEKM*. The entries of the *VEKM* are not restricted in any way whatsoever other than being in s-o-p form, though it may be advisable to have each of them in a minimal form. The concept of entered loops is new and fits the use of variable entries. Let a prime implicant $P$ (or even any unrestricted term or product ) be given by

$$P = \bigcap_{i \in E} Y_i \bigcap_{i \in M} Y_i, \tag{12}$$

where $E$ and $M$ are two disjoint sets of indices representing, respectively, subsets of the two sets of indices for the entered and map variables, and $Y_i$ stands for a literal of the variable $X_i$, i.e., $Y_i$ is either $X_i$ or $\bar{X}_i$. Now, rewrite $P$ as

$$P = P_E \; P_M, \tag{13}$$

where

$$P_E = \bigcap_{i \in E} Y_i, \tag{14a}$$

$$P_M = \bigcap_{i \in M} Y_i, \tag{14b}$$

are to be called the loop depth and loop coverage of the product $P$, respectively. Again, rewrite (13) as an expression of a ratio or residue

$$P_M = P/P_E. \tag{15}$$

Note that the transformation from (13) to (15) is possible because the sets of literals in the product $P_E$ are disjoint from those in the product $P_M$.

Now, it is possible to have a *VEKM* representation of the given $P$ by a loop whose product is $P_M$ which consists solely of map variables. However, such a loop is different from a traditional *CKM* loop, since it does not generally cover 1 entries in the map. Instead, such a loop has a coverage dictated by its depth $P_E$ which is a product of entered variables only. Therefore, the loop $P_M = P/P_E$ covers only entered terms that are subsumed by $P_E$ ( including $P_E$ itself, of course).

With the concept of *VEKM* loops so defined, it is now a straightforward task to continue. We draw a *VEKM* loop for every *PI* of the given *ISSF* and then write a Petrick function by considering the coverage of each asserted term in each cell of the *VEKM*. Note that the present situation differs slightly from that of section II since we now cover entered terms of the *VEKM* rather than the column keys or coordinates of the *VECM*. In the present case, the Petrick function can still be expressed via (4) but $A_j$ therein is now understood to mean an expression of the possible ways of selecting prime implicant loops to cover a particular asserted entered term $T_j$. Since such a product $T_j$ may appear in several *VEKM* cells $l$, the following p-o-s expression for $A_j$ can be written

$$A_j = \bigcap_l A_{jl}, \tag{16}$$

where $A_{jl}$ is an expression of all the possible ways of selecting *PI* loops to cover the asserted entered term $T_j$ in cell $l$ of the *VEKM*, and the ANDing in (16) runs over all such cells in which $T_j$ appears. Equations (4) and (16) can be combined to yield the Petrick function

$$PF = \bigcap_{j=1}^{j_{max}} \bigcap_l A_{jl}, \tag{17}$$

where the alterm $A_{jl}$ is given by the right hand side of (5) with the union operator therein running over all sets $I$ that satisfy (6a) together with the requirement

$$\bigcup_{i \in I} \left( P_i / m_l \right) \geq T_j \quad \text{(irr.)}, \tag{18}$$

or equivalently

$$\bigcup_{i \in I} \left( P_{i\,E} \right)_l \geq T_j \quad \text{(irr.)}. \tag{19}$$

In (18) $m_l$ stands for the minterm expansion of cell $l$ and in (19) and $\left(P_{i\,E}\right)_l = \left(P_i / P_{iM}\right)_l$ is the entered part of prime implicant loop $P_l$ in cell $l$. The present procedure is further clarified with an example.

**Example 4**
Consider the *ISSF* $f$ given in Example 2 of [6], namely

$$\begin{aligned} f = {} & \bar{A}B\bar{E} \vee A\bar{B}E \vee \bar{A}D\bar{E} \vee \bar{A}C\bar{E} \vee B\bar{C}D\bar{E} \vee AC\bar{D}E \vee \\ & d\left(\bar{A}E \vee A\bar{B}C\bar{E} \vee AB\bar{C}E \vee \bar{B}\,\bar{C}DE\right) \end{aligned} \tag{20}$$

Figure 1 is a replica of the *VEKM* representation for $f$ given in Fig. 4 in [6], with prime implicant loops added and don't care entries removed. Table 4 lists all asserted entered implicants $T_j$ and the cells in which they appear together with the corresponding expressions of $A_{jl}$. Finally, the Petrick function is written as

$$\begin{aligned} PF &= (P_1 \vee P_4)P_5P_3P_2(P_7 \vee P_1 P_8)P_9 \\ &= P_1P_2P_3P_5P_8P_9 \vee P_1P_2P_3P_5P_7P_9 \vee P_2P_3P_4P_5P_7P_9, \end{aligned} \tag{21}$$

which means that the *ISSF* considered has 3 irredundant disjunctive forms which turn out to be minimal ( in agreement with our findings in Example 2 of [6]).

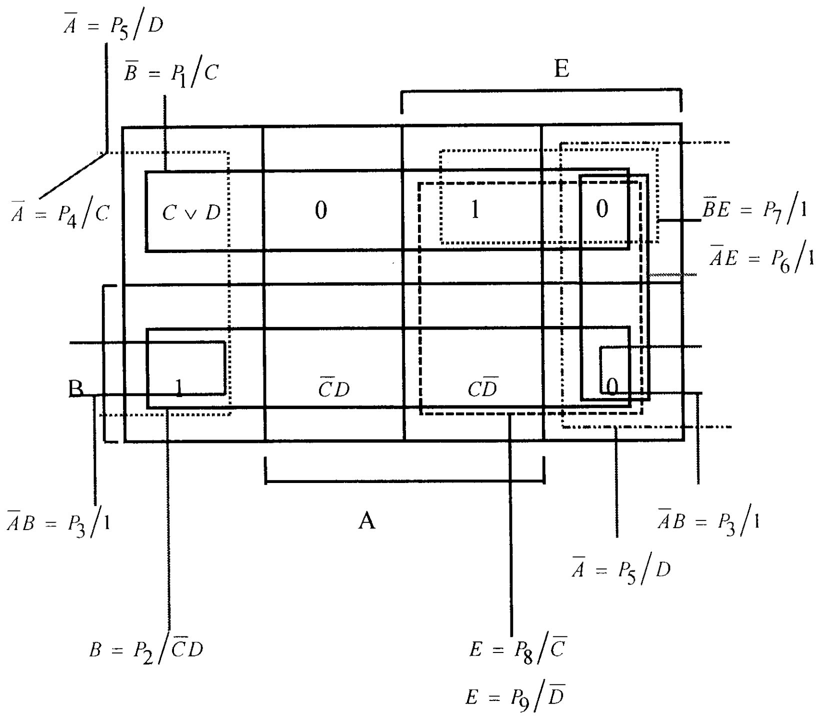


**Fig 1. Prime implicant loops drawn on a *VEKM* to cover the asserted part of an *ISSF*.**

## 4. Use of the *Vekm* as a Cover Map

This section presents an improved version of the method in section III that avoids the actual drawing of prime implicant loops but instead uses the *VEKM* as a cover map. In each *VEKM* cell, two different entities are shown, namely, the asserted entry of the cell and a list of prime implicants that are candidates for covering it. Cells with 0 entries are left blank. For each prime implicant loop $P_i = P_{iM}\ P_{iE,}$ we conceptually define the loop domain or coverage as that defined by its map product $P_{iM}$ without actually drawing such a loop. Then, we enter that *PI* information in the form

$(P_i$ : literals of $P_{iE})$ in each cell of its domain, only wherever this is deemed necessary, i.e., only wherever there is an entry to be covered in the concerned cell. It is now straightforward to construct the Petrick function by using the method described in the section III. The present method has the visual benefit that the *VEKM* is not littered with unnecessary details such as 0 entries or too many overlapping loops. This visual merit can be used to advantage, as it allows the application of the method to larger and/or more complicated problems as can be seen from the following examples.

**Example 5** (Example 4 revisited)

Figure 2 shows the *VEKM* in Fig. 1 redrawn as a cover matrix. The information in Table 4 and hence the Petrick function in (21) are more readily obtained in this case thanks to the fact that each *VEKM* cell is now entered by both the asserted terms to be covered and the *PIs* that can be used to cover them.

**Table 4. Pertaining to the solution of Example 4**

| Asserted entered Implicant $T_j$ | $C$ | $D$ | 1 | $\bar{C}D$ | 1 | $C\bar{D}$ |
|---|---|---|---|---|---|---|
| Cell $l$ | $\bar{E}\bar{A}\bar{B}$ | $\bar{E}\bar{A}\bar{B}$ | $\bar{E}\bar{A}B$ | $\bar{E}AB$ | $EA\bar{B}$ | $EAB$ |
| Ways of covering $A_{jl}$ | $P_1 \vee P_4$ | $P_5$ | $P_3$ | $P_2$ | $P_7 \vee P_1 P_8$ | $P_9$ |

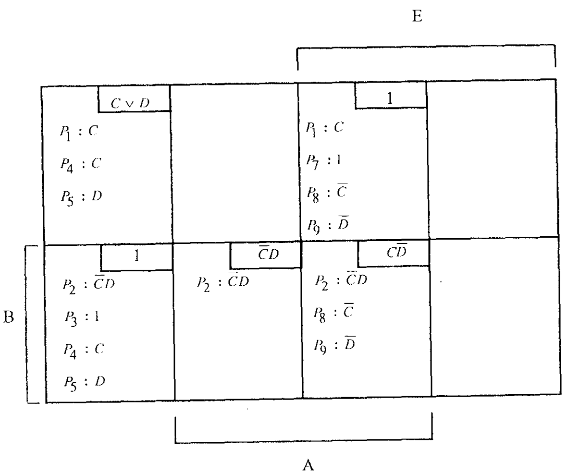


**Fig. 2. The VERM in Fig. 1 redrawn as a cover map. In each cell, we show its asserted entry and also the prime implicants that are candidates for covering it. Cells with 0 entries are left blank.**

**Example 6** ( Example 1 revisited)

Figure 3 shows a *VEKM* (used as a cover map) for the *ISSF* in Example 1. Note that this *ISSF* has 16 *Pis* and it would be difficult to draw *VEKM* loops for this large number of *PIs*. However, it is still manageable to present such *PIs* as covering candidates by the present method. The Petrick function is immediately written as

$$\begin{aligned} PF &= (P_3 \vee P_5)(P_1 \vee P_5 \vee P_6)(P_1)(P_2 \vee P_3)(P_2)(P_7 \vee P_8)(P_6)(P_4) \\ &= P_1\, P_2\, P_4\, P_6\, (P_3 \vee P_5)(P_7 \vee P_8), \end{aligned} \tag{22}$$

Note that the *PF* in (22) is equivalent to that in (8) though it looks initially different from it.

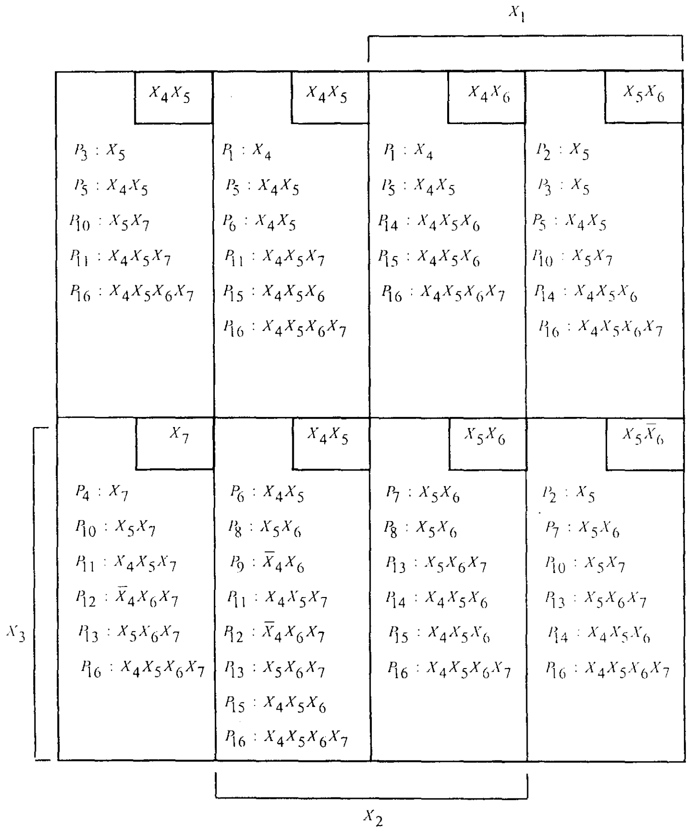


**Fig. 3. A VEKM used as a cover map for the asserted part of the ISS in Example 1.**

## 5. Conclusions

This paper has presented algorithmic *VEKM* or *VEKM*-related procedures to solve the classical covering problem, i.e., to obtain all the irredundant disjunctive forms of a switching function $f$ and hence its exactly minimal sums when its complete sum is available. Dual versions of these procedures can be used to start with the complete product of $f$ and end with all its irredundant conjunctive forms and its minimal product(s). Details of these dual versions can be worked out if one follows the duality rules in [6].

Many of the existing methods for handling the covering problem are easily adaptable to *VEKM* implementations. Notable among these are two methods described by Brown [1], the former of which is based on syllogistic reasoning [1, pp. 146-148], while the latter entails repeated tautology tests [1, pp. 117-118]. Also attractive is the method due to Hammer and Rudeanu [12, pp. 285-294] which converts the present covering problem into one of minimizing a linear pseudo-Boolean function subject to linear pseudo-Boolean constraints.

## References


[1] Brown, F.M. *Boolean Reasoning: The Logic of Boolean Equations*. Boston: Kluwer Academic Publishers, 1990.

[2] Gregg, J.R. *Ones and Zeros: Understanding Boolean Algebra, Digital Circuits, and the Logic of Sets*. New York: IEEE Press, 1998.

[3] Muroga, S. *Logic Design and Switching Theory*. New York: Wiley, 1979.

[4] Rushdi, A.M. and Al-Yahya, H. A. "Derivation of the Complete Sum of a Switching Function with the Aid of the Variable-Entered Karnaugh Map." *Journal of King Saud University* (Engineering Sciences).

[5] Reusch, B. "Generation of Prime Implicants from Subfunctions and a Unifying Approach to the Covering Problem." *IEEE Trans. Computers,* C-24 (1975), 924-930.

[6] Rushdi, A.M. "Improved Variable-Entered Karnaugh Map Procedures." *Computers and Electrical Engineering*, 13 (1987), 41- 52.

[7] Quine, W. V. "A Way to Simplify Truth Functions." *Amer. Math. Mon.*, 62 (1956), 627-631.

[8] McCluskey, E. J. "Minimization of Boolean Functions." *Bell Syst. J.*, 35 (1956), 1417-1444.

[9] Ghazala, M.J. "Irredundant Disjunctive and Conjunctive Forms of a Boolean Function." *IBM J. Res. Develop.*, 1 (1956), 171-176.

[10] Chang, D.M.Y. and Mott, T.H. "Computing Irredundant Normal Forms from Abbreviated Presence Functions." *IEEE Trans. Electron. Comput.*, EC-14 (1965), 335-342.

[11] Thayse, A. "Meet and Join Derivatives and Their Use in Switching Theory." *IEEE Trans. Computers,* C-27 (1978), 713-720.

[12] Hammer, P.L. and Rudeanu, S. *Boolean Methods in Operatios Research and Related Areas*. Heidelberg, Germany: Springer Verlag, 1968

# طرق استخدام خريطة كارنوه متغيرة المحتويات للحصول على الصيغ غير الوافرة لمجموع المضروبات لدالة تبديلية من مجموعها الكامل

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**ملخص البحث**. إن مسألة التغطية ، وهي مسألة تقليدية في نظرية التبديل ، تتعلق باختيار غير وافر لبعض الضامنات الأولية التي تغطي الجزء المؤكد من دالـة تبديليـة غـير كاملـة التحديـد. تتبنـى ورقـة البحـث هـذه اقتراح استخدام خريطة كارنوه متغيرة المحتويات في حل هذه المسألة. ولتهيئة الأمور لبلوغ هـذا الهـدف يتـم تقديم استعراض توضيحي مبتكر لمصفوفة التغطية متغيرة المحتويات وهي مصفوفـة وثيقـة الصلـة بالخريطـة المذكورة. ومن ثـم تقـدم الورقـة حلـين لمسـألة التغطيـة يستعملان هـذه الخريطـة. يعتمـد الحـل الأول علـى مفهوم جديد هو مفهوم الحلقات في الخريطة متغيرة المحتويات. وبينما تتميز الحلقة في خريطة كارنوه العادية بمضروب واحد ، فإن الحلقة في الخريطة متغيرة المحتويات تعرَّف بمضروبين يسمى أولها تغطية الحلقـة( وهـو يعتمد فقط على متغيرات الحلقة) ويسمى الثاني عمق الحلقة (وهو يعتمد فقط على المتغيرات المدخلة). أما في الحل الثاني فتستعمل الخريطة متغيرة المحتويات مباشـرة كخريطـة تغطيـة ، ويتـم ذلـك بطريقـة لاتشـوش الخريطة بتفصيلات مبعثرة غير ضرورية أو بحلقات متداخلة عديدة. تتضمن الورقـة بيانـا لجميـع إجـراءات الحل من خلال أمثلة توضيحية تساعد على الكشف عن مزاياها التصويرية وكذلك إمكانية تطبيقـها علـى مسائل كبيرة نسبيا.