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Analysis of the Dynamic Response of Cross-ply Laminated Shallow Shells with Various Boundary Conditions

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Abstract. Analytical solutions of the dynamic response of the classical, first-order and third-order theories of cross-ply laminated shallow shells are developed for various boundary conditions. The solutions are applicable to laminated shells with two opposite edges simply supported and the remaining ones can have arbitrary combinations of free, clamped and simply supported boundary conditions. A Levy type method in conjunction with generalized modal approach is used to obtain these solutions. For thick shells, the classical shell theory predicts deflections and stresses significantly different from those of the third-order theory. The third-order theory and first-order theory results are very close to each other for response and normal stress. However, the third-order theory does not require the use of shear correction factors.

Introduction

The analysis of laminated composite shells has been the subject of significant research interest in recent years. The classical lamination shell theories based on the Love-Kirchhoff assumptions are adequate to predict the gross behavior of thin laminates. A survey of different classical lamination theories can be found in [1-2]. When the structures are rather thick or when they exhibit high anisotropy ratios, the transverse shear deformation effect has to be incorporated. In such cases more refined theories are needed. Numerous first-order and higher-order shear deformation theories of laminated composite shells are presented in the literature [3-11]. The third-order theory used in the present study is proposed by Reddy and Liu [5], in which the surface displacements are expanded up to the cubic term in thickness coordinate while the transverse deflection is assumed to be constant through the thickness. The nine undetermined functions are reduced to five by imposing stress-free boundary conditions on the transverse shear stress on the bounding surfaces of the shell. Since the theory accounts for parabolic distribution of the transverse shear stresses, no shear correction coefficients are required.

Closed-form solutions for the dynamic response of laminated plates and shells have been developed mainly for the case of simply supported boundary condition [12-19]. Analytical solutions for the dynamic response of composite laminates for a variety of boundary conditions are developed in [20-24], where in [24], a brief note is introduced about shallow shells. Ritz, Galerkin and other approximate methods are used for general boundary conditions and lamination schemes.

In the present work, a generalized modal approach in conjunction with Levy method is presented to solve for the transient response of cross-ply laminated shallow shells with various boundary conditions and for arbitrary loadings. The equations of motions of the classical, first-order and third-order theories are converted into a single-order system of equation by using state variables. The biorthogonality conditions of principal modes of the original and adjoint eigenfunctions are used to decouple the state space equation. To demonstrate the method, numerical results of the three theories for center deflections and stresses of spherical shells subjected to sinusoidal loading in spatial domain and sine pulse loading in time domain are presented.

Equations of Motion

The higher-order shear deformation theory (HSDT) used in the present study is based on the following displacement field (see Reddy and Liu [5]):

$$\bar{u} = (1+\frac{\zeta}{R_1})u + \zeta\phi_1 - n_2\zeta^3(\phi_1 + \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1})$$

$$\bar{v} = (1+\frac{\zeta}{R_2})v + \zeta\phi_2 - n_2\zeta^3(\phi_2 + \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2})$$

$$\bar{w} = w \tag{1}$$

where $\bar{u}, \bar{v}, \bar{w}$ are the displacements along the orthogonal curvilinear coordinates such that the ξ_1 and ξ_2 -curves are lines of principal curvature on the midsurface $\zeta = 0$, and ζ-curves are straight lines perpendicular to the surface $\zeta = 0$, (u,v,w) are the displacements of a point on the middle surface, and ϕ_1 and ϕ_2 are the rotations at $\zeta = 0$ of normals to the mid surface with respect to the ξ_2 and ξ_1 -axes, respectively. The parameters R_1 and R_2 denote the values of the principal radii of curvature of the middle surface (see Fig. 1), and α_1 and α_2 are the surface metrics. All displacement components $(u, v, w, \phi_1, \phi_2)$ are functions of (ξ_1, ξ_2) and time t.

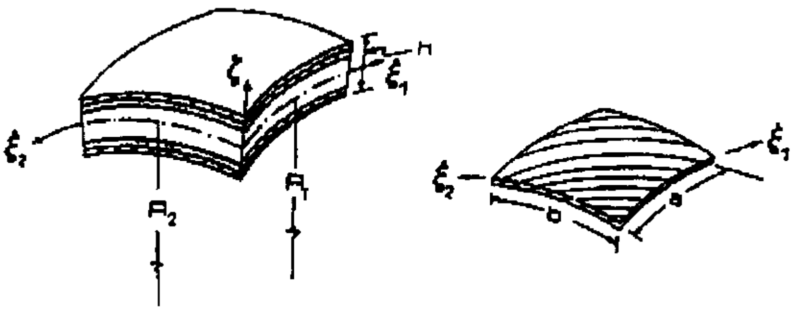


Fig. 1. Geometry and coordinate system of a double curved shell panel.

Substituting Eq. (1) into the linear strain-displacement relations of a shell referred to an orthogonal curvilinear coordinate system, we obtain

$$\begin{aligned}\varepsilon_1 &= \varepsilon_1^0+\zeta(\kappa_1^0+\gamma\zeta^2\kappa_1^2)\\ \varepsilon_2 &= \varepsilon_2^0+\zeta(\kappa_2^0+\gamma\zeta^2\kappa_2^2)\\ \varepsilon_4 &= \varepsilon_4^0+\gamma\zeta^2\kappa_4^1\\ \varepsilon_5 &= \varepsilon_5^0+\gamma\zeta^2\kappa_5^1\\ \varepsilon_6 &= \varepsilon_6^0+\zeta(\kappa_6^0+\gamma\zeta^2\kappa_6^2),\end{aligned} \tag{2}$$

where

$$\begin{aligned}&\varepsilon_1^0 = \frac{\partial u}{\partial x_1}+\frac{w}{R_1},\quad \kappa_1^0 = \frac{\partial \phi_1}{\partial x_1},\quad \kappa_1^2 = -n_2\left(\frac{\partial \phi_1}{\partial x_1}+\frac{\partial^2 w}{\partial x_1^2}\right),\\ &\varepsilon_2^0 = \frac{\partial v}{\partial x_2}+\frac{w}{R_2},\quad \kappa_2^0 = \frac{\partial \phi_2}{\partial x_2},\quad \kappa_2^2 = -n_2\left(\frac{\partial \phi_2}{\partial x_2}+\frac{\partial^2 w}{\partial x_2^2}\right),\\ &\varepsilon_4^0 = \phi_2+\frac{\partial w}{\partial x_2},\quad \kappa_4^1 = -n_1\left(\phi_2+\frac{\partial w}{\partial x_2}\right),\\ &\varepsilon_5^0 = \phi_1+\frac{\partial w}{\partial x_1},\quad \kappa_5^1 = -n_1\left(\phi_1+\frac{\partial w}{\partial x_1}\right),\\ &\varepsilon_6^0 = \frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2},\quad \kappa_6^0 = \frac{\partial \phi_2}{\partial x_1}+\frac{\partial \phi_1}{\partial x_2},\quad \kappa_6^2 = -n_2\left(\frac{\partial \phi_2}{\partial x_1}+\frac{\partial \phi_1}{\partial x_2}+2\frac{\partial^2 w}{\partial x_1\partial x_2}\right).\end{aligned} \tag{3}$$

Here x_i denote the cartesian coordinates $(dx_i=\alpha_i\,d\xi_i,\ i=1,2)$, and $n_1 = 4/h^2$ and $n_2 = n_1/3$.

The stress-strain relations for the kth lamina are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix}_{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ & Q_{22} & Q_{26} & 0 & 0 \\ & & Q_{66} & 0 & 0 \\ & \text{symm.} & & Q_{44} & 0 \\ & & & & Q_{55} \end{bmatrix}_{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{4}$$

Where σ_1 ,σ_2 are normal components; and $\sigma_6, \sigma_4, \sigma_5$ are shear stress components (see Fig.2); and $Q_{ij}^{(k)}$ are the material constants of the kth lamina in the laminate coordinate system.

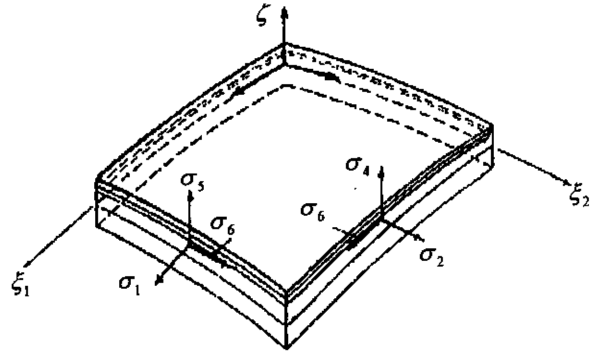


Fig. 2. Stress components in shell coordinates.

Using Hamilton's principle, the equations of motion appropriate for the displacement field (1) and the constitutive equations (4) are derived in [5] as:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = \bar{I}_1 \ddot{u} + \bar{I}_2 \ddot{\phi}_1 - \gamma \bar{I}_3 \frac{\partial \ddot{w}}{\partial x_1}$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = \bar{I}_1' \ddot{v} + \bar{I}_2' \ddot{\phi}_2 - \gamma \bar{I}_3' \frac{\partial \ddot{w}}{\partial x_2}$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \gamma n_1 \left(\frac{\partial K_1}{\partial x_1} + \frac{\partial K_2}{\partial x_2}\right) + \gamma n_2 \left(\frac{\partial^2 P_1}{\partial x_1^2} + \frac{\partial^2 P_2}{\partial x_2^2} + 2\frac{\partial^2 P_6}{\partial x_1 \partial x_2}\right) - \frac{N_1}{R_1} - \frac{N_2}{R_2} + q =$$

$$I_1 \ddot{w} + \gamma [\bar{I}_3 \frac{\partial \ddot{u}}{\partial x_1} + \bar{I}_5 \frac{\partial \ddot{\phi}_1}{\partial x_1} + \bar{I}_3' \frac{\partial \ddot{v}}{\partial x_2} + \bar{I}_5' \frac{\partial \ddot{\phi}_2}{\partial x_2} - n_2^2 I_7 \left(\frac{\partial^2 \ddot{w}}{\partial x_1^2} + \frac{\partial^2 \ddot{w}}{\partial x_2^2}\right)]$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 + \gamma n_1 K_1 - \gamma n_2 \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_6}{\partial x_2}\right) = \bar{I}_2 \ddot{u} + \bar{I}_4 \ddot{\phi}_1 - \gamma \bar{I}_5 \frac{\partial \ddot{w}}{\partial x_1}$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - Q_2 + \gamma n_1 K_2 - \gamma n_2 \left(\frac{\partial P_6}{\partial x_1} + \frac{\partial P_2}{\partial x_2}\right) = \bar{I}_2' \ddot{v} + \bar{I}_4' \ddot{\phi}_2 - \gamma \bar{I}_5' \frac{\partial \ddot{w}}{\partial x_2} \tag{5}$$

where superposed dot denotes differentiation with respect to time, q is the distributed transverse load which is the only external load considered in the analysis, and N_i, M_i, ... are the stress resultants:

$$(N_i, M_i, P_i) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_i^{(k)}(1, \zeta, \zeta^3)\, d\zeta \quad (i=1,2,6)$$

$$(Q_1, K_1) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_5^{(k)}(1, \zeta^2)\, d\zeta$$

$$(Q_2, K_2) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \sigma_4^{(k)}(1, \zeta^2)\, d\zeta. \tag{6}$$

The inertias are defined by the equations,

$$\overline{I_1} = I_1 + \gamma \frac{2}{R_1} I_2 \,,$$

$$\overline{I_2} = I_2 + \gamma \frac{1}{R_1} I_3 - \gamma n_2 I_4 - \gamma \frac{n_2}{R_1} I_5 \,,$$

$$\overline{I_3} = n_2 I_4 + \frac{n_2}{R_1} I_5 \,,$$

$$\overline{I_4} = I_3 - 2\gamma n_2 I_5 + \gamma n_2^2 I_7 \,,$$

$$\overline{I_5} = n_2 I_5 - n_2^2 I_7 \,,$$

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^{N} \int_{\zeta_{k-1}}^{\zeta_k} \rho^{(k)}(1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6)\, d\zeta \,, \tag{7}$$

Where $\rho^{(k)}$ is the mass density of the lamina per unit volume and $\bar{I}_i'$ are the same as $\bar{I}_i$ except that R_1 is replaced by R_2 . The resultants are related to the total strains by

$$N_i = A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j^0 + \gamma E_{ij}\kappa_j^2$$

$$M_i = B_{ij}\varepsilon_j^0 + D_{ij}\kappa_j^0 + \gamma F_{ij}\kappa_j^2$$

$$P_i = E_{ij}\varepsilon_j^0 + F_{ij}\kappa_j^0 + H_{ij}\kappa_j^2 \tag{8}$$

for i,j = 1,2,6

Substitution of Eqs. (13) and (14) into the equations of motion of the three theories results in the following systems of equations

HSDT

$$\begin{aligned}
U_m'' &= c_1 U_m + c_2 V_m' + c_3 W_m' + c_4 W_m''' + c_5 \Phi_{1m} + c_6 \Phi_{2m}' \\
&+ b_1 \ddot{U}_m + b_2 \ddot{W}_m' + b_3 \ddot{\Phi}_{1m} \\
V_m'' &= c_7 U_m' + c_8 V_m + c_9 W_m + c_{10} W_m'' + c_{11} \Phi_{1m}' + c_{12} \Phi_{2m} \\
&+ b_4 \ddot{V}_m + b_5 \ddot{W}_m + b_6 \ddot{\Phi}_{2m} \\
W_m'''' &= c_{13} U_m' + c_{14} V_m + c_{15} W_m + c_{16} W_m'' + c_{17} \Phi_{1m}' + c_{18} \Phi_{2m} \\
&+ b_7 \ddot{U}_m' + b_8 \ddot{V}_m + b_9 \ddot{W}_m + b_{10} \ddot{W}_m'' + b_{11} \ddot{\Phi}_{1m}' + b_{12} \ddot{\Phi}_{2m} + a_0 f_m \\
\Phi_{1m}'' &= c_{19} U_m + c_{20} V_m' + c_{21} W_m' + c_{22} W_m''' + c_{23} \Phi_{1m} + c_{24} \Phi_{2m}' \\
&+ b_{13} \ddot{U}_m + b_{14} \ddot{W}_m' + b_{15} \ddot{\Phi}_{1m} \\
\Phi_{2m}'' &= c_{25} U_m' + c_{26} V_m + c_{27} W_m + c_{28} W_m'' + c_{29} \Phi_{1m}' + c_{30} \Phi_{2m} \\
&+ b_{16} \ddot{V}_m + b_{17} \ddot{W}_m + b_{18} \ddot{\Phi}_{2m}
\end{aligned} \tag{15}$$

FSDT

$$\begin{aligned}
U_m'' &= c_1 U_m + c_2 V_m' + c_3 W_m' + c_4 \Phi_{1m} + c_5 \Phi_{2m}' + b_1 \ddot{U}_m + b_2 \ddot{\Phi}_{1m} \\
V_m'' &= c_6 U_m' + c_7 V_m + c_8 W_m + c_9 \Phi_{1m}' + c_{10} \Phi_{2m} + b_3 \ddot{V}_m + b_4 \ddot{\Phi}_{2m} \\
W_m'' &= c_{11} U_m' + c_{12} V_m + c_{13} W_m + c_{14} \Phi_{1m}' + c_{15} \Phi_{2m} + b_5 \ddot{W}_m + a_0 f_m \\
\Phi_{1m}'' &= c_{16} U_m + c_{17} V_m' + c_{18} W_m' + c_{19} \Phi_{1m} + c_{20} \Phi_{2m}' + b_6 \ddot{U}_m + b_7 \ddot{\Phi}_{1m} \\
\Phi_{2m}'' &= c_{21} U_m' + c_{22} V_m + c_{23} W_m + c_{24} \Phi_{1m}' + c_{25} \Phi_{2m} + b_8 \ddot{V}_m + b_9 \ddot{\Phi}_{2m}
\end{aligned} \tag{16}$$

CST

$$\begin{aligned}
U_m'' &= c_1 U_m + c_2 V_m' + c_3 W_m' + c_4 W_m''' + b_1 \ddot{U}_m + b_2 \ddot{W}_m' \\
V_m'' &= c_5 U_m' + c_6 V_m + c_7 W_m + c_8 W_m'' + b_3 \ddot{V}_m + b_4 \ddot{W}_m \\
W_m'''' &= c_9 U_m' + c_{10} V_m + c_{11} W_m + c_{12} W_m'' + b_5 \ddot{U}_m' + b_6 \ddot{V}_m \\
&+ b_7 \ddot{W}_m + b_8 \ddot{W}_m'' + a_0 f_m
\end{aligned} \tag{17}$$

where a prime and dot on a quantity denote the derivative with respect to x_1 and time t, respectively. The coefficients in Eqs. (15), (16) and (17) are presented in Appendix A.

In order to reduce the system of Eqs. (15), (16) and (17) to a state form, the components of the state vector $\{ y(x_1,t) \}$ associated with each theory are defined as

HSDT

$$y_{1m} = W_m, y_{2m} = W_m', y_{3m} = W_m'', y_{4m} = U_m, y_{5m} = V_m, y_{6m} = \Phi_{1m},$$

$$y_{7m} = \Phi_{2m}, y_{8m} = W_m''', y_{9m} = U_m', y_{10m} = V_m', y_{11m} = \Phi_{1m}', y_{12m} = \Phi_{2m}' \tag{18}$$

FSDT

$$\begin{aligned} y_{1m} &= W_m, y_{2m} = U_m, y_{3m} = V_m, y_{4m} = \Phi_{1m}, y_{5m} = \Phi_{2m} \\ y_{6m} &= W_m', y_{7m} = U_m', y_{8m} = V_m', y_{9m} = \Phi_{1m}', y_{10m} = \Phi_{2m}' \end{aligned} \tag{19}$$

CST

$$\begin{aligned} y_{1m} &= W_m, y_{2m} = W_m', y_{3m} = W_m'', y_{4m} = U_m \\ y_{5m} &= V_m, y_{6m} = W_m''', y_{7m} = U_m', y_{8m} = V_m' \end{aligned} \tag{20}$$

Using (18), (19) and (20), the system of equations (15), (16) and (17) may be expressed in the form

$$\{ y' \} = [M]\{ \ddot{y} \} + [K]\{ y \} + \{ r \} \tag{21}$$

where the matrices [M] and [K] are defined in Appendix B for HSDT, FSDT and CST as (12×12), (10×10) and (8×8) matrices, respectively. The elements of the load vector {r} are

$$\{ r \}^T = \{ 0,0,0,0,0,0,0,a_0\, f_m,0,0,0,0 \} \quad \text{for HSDT} \tag{22}$$

$$\{ r \}^T = \{ 0,0,0,0,0,a_0\, f_m,0,0,0,0 \} \quad \text{for FSDT} \tag{23}$$

$$\{ r \}^T = \{ 0,0,0,0,0,a_0\, f_m,0,0 \} \quad \text{for CST} \tag{24}$$

Free Vibration Problem

In the case of free vibration problem, the vector { y } will be separated into time and spatial coordinates as :

$$\{ y \} = \{ Y_m(x_1) \} T_m(t) \tag{25}$$

To obtain the frequencies and the corresponding eigenfunctions, the generalized coordinates

$T_m(t)$ must satisfy

$$\ddot{T}_m + \omega_m^2 T_m = 0 \tag{26}$$

and the eigenfunctions $\{ Y_m \}$ will fulfil the following equation

$$\{ Y' \} = [A] \{ Y \} \tag{27}$$

where

$$[A] = [K] - \omega_m^2 [M] \tag{28}$$

ω_m is the natural frequency corresponding to the mth mode. There are infinite frequencies for each value of m and the dynamic response is governed mainly by the fundamental frequency of each mode.

The solution to Eq. (27) is given by:

$$\{ Y(x_1) \} = [D] [\eta(x_1)] \{ l \} \tag{29}$$

where

$$\{ l \} = [D]^{-1} \{ k \} \tag{30}$$

and

$$[\eta(x_1)] = \begin{bmatrix} e^{\lambda_1 x_1} & & \underline{0} \\ & \vdots & \\ & \vdots & \\ \underline{0} & & e^{\lambda_n x_1} \end{bmatrix} \tag{31}$$

where n = 12 for HSDT, n = 10 for FSDT and n = 8 for CST, and λ_i are the distinct eigen values of the matrix [A] while [D] denotes the matrix of eigen vectors of [A]. Substitution of (29) into the boundary conditions associated with the edges $x_1 = \pm a / 2$ results in a set of homogeneous algebraic equations of the form

$$[B] [D]^{-1} \{ k \} = \{ 0 \} \tag{32}$$

For nontrivial solution of Eq. (32), the determinant should be zero

$$|B| / |D| = 0 \tag{33}$$

Equations (33) and (29) give the eigen frequencies and the associated eigen functions, respectively. The boundary conditions for simply supported (S), clamped (C) and free (F) at the edges $x_1 = \pm a/2$ for the three theories are:

HSDT

$$S: v = w = \phi_2 = N_1 = M_1 = P_1 = 0$$

$$C: u = v = w = \phi_1 = \phi_2 = \frac{\partial w}{\partial x_1} = 0$$

$$F: N_1 = M_1 = P_1 = N_6 = M_6 - n_2 P_6 = Q_1 - n_1 K_1 + n_2 \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_6}{\partial x_2}\right) = 0 \tag{34}$$

FSDT

$$S: v = w = \phi_2 = N_1 = M_1 = 0$$

$$C: u = v = w = \phi_1 = \phi_2 = 0$$

$$F: N_1 = M_1 = Q_1 = N_6 = M_6 = 0 \tag{35}$$

CST

$$S: v = w = N_1 = M_1 = 0$$

$$C: u = v = w = \frac{\partial w}{\partial x_1} = 0$$

$$F: N_1 = M_1 = N_6 + \frac{1}{R_2} M_6 = \frac{\partial M_1}{\partial x_1} + 2\frac{\partial M_6}{\partial x_2} = 0 \tag{36}$$

Adjoint Problem

Equation (21) is not a self-adjoint equation and the eigenfunctions do not form an orthogonal set, therefore, we must obtain the eigenfunction of the adjoint of Eq. (27) in order to decouple Eq. (21). Nayfeh in his book [25] showed that the adjoint of Eq. (27) is:

$$\{Z'\} = -[A]^T \{Z\} \tag{37}$$

with the following associated boundary conditions

$$\{Z\}^T \{Y\} \Big|_{-a/2}^{a/2} = 0 \tag{38}$$

According to Eq. (38), the following boundary conditions will be defined at the edges $x_1 = \pm a/2$:

HSDT

$$
\begin{aligned}
&S: Z_2 = Z_4 = Z_6 = Z_8 = Z_{10} = Z_{12} = 0\\
&C: Z_3 = Z_8 = Z_9 = Z_{10} = Z_{11} = Z_{12} = 0\\
&F: d_1 Z_1 + d_4 Z_3 + d_7 Z_5 + Z_7 = 0\\
&d_2 Z_1 + d_5 Z_3 + d_8 Z_5 + Z_9 = 0\\
&d_3 Z_1 + d_6 Z_3 + d_9 Z_5 + Z_{11} = 0\\
&d_{10} Z_2 + d_{13} Z_4 + d_{16} Z_6 + Z_8 = 0\\
&d_{11} Z_2 + d_{14} Z_4 + d_{17} Z_6 + Z_{10} = 0\\
&d_{12} Z_2 + d_{15} Z_4 + d_{18} Z_6 + Z_{12} = 0
\end{aligned} \tag{39}
$$

FSDT

$$
\begin{aligned}
&S: Z_2 = Z_4 = Z_6 = Z_8 = Z_{10} = 0\\
&C: Z_6 = Z_7 = Z_8 = Z_9 = Z_{10} = 0\\
&F: d_1 Z_7 + d_4 Z_9 + Z_1 = 0\\
&d_2 Z_7 + d_5 Z_9 + Z_3 = 0\\
&d_3 Z_7 + d_6 Z_9 + Z_5 = 0\\
&Z_2 - \beta Z_8 = 0\\
&Z_4 - Z_6 - \beta Z_{10} = 0
\end{aligned} \tag{40}
$$

CST

$$
\begin{aligned}
&S: Z_2 = Z_4 = Z_6 = Z_8 = 0\\
&C: Z_3 = Z_6 = Z_7 = Z_8 = 0\\
&F: d_1 Z_1 + d_3 Z_3 + Z_5 = 0\\
&d_2 Z_1 + d_4 Z_3 + Z_7 = 0\\
&d_5 Z_2 + d_7 Z_4 + Z_6 = 0\\
&d_6 Z_2 + d_8 Z_4 + Z_8 = 0
\end{aligned} \tag{41}
$$

The constants (d_i) in equations (39), (40) and (41) are presented in Appendix C.

A formal solution of equation (37) is given by

$$\{Z\} = [C][\xi(x_1)]\{n\} \tag{42}$$

$$[\xi(x_1)] = \begin{bmatrix} e^{-\lambda_1 x_1} & & \underline{0} \\ & \vdots & \\ & \vdots & \\ \underline{0} & & e^{-\lambda_n x_1} \end{bmatrix} \tag{43}$$

where [C] denotes the matrix of eigenvectors of $-[A]^T$.

Substitution of Eq. (42) into the corresponding boundary conditions defined in Eqs. (39-41) for the three theories at the edges $x_1 = \pm a/2$ results a homogeneous algebraic equations of the form

$$[E]\{n\} = 0 \tag{44}$$

we have to solve for the eigenvector $\{n\}$ corresponding to each frequency ω.

Dynamic Response

Making use of the following biorthogonality conditions of the natural modes with respect to the eigenfunctions $\{Y_m\}$ and $\{Z_n\}$,

$$-\int_{-a/2}^{a/2} \{Z_n\}^T [M]\{Y_m\} dx_1 = M_m \delta_{mn} \tag{45}$$

$$\int_{-a/2}^{a/2} \{Z_n\}^T (\{Y_m'\} - [K]\{Y_m\}) dx_1 = \omega_m^2 M_m \delta_{mn} \tag{46}$$

and substituting Eq. (25) in Eq. (21) and left multiplication by the adjoint eigenfunction $\{Z_n\}^T$ and integrate over the domain, we obtain

$$\ddot{T}_m(t) + \omega_m^2 T_m(t) = \frac{1}{M_m} \int_{-a/2}^{a/2} \{Z_m\}^T \{r_m\} dx_1 \tag{47}$$

For zero initial conditions, the state vector $\{y\}$ will be expressed as

$$\{y_m(x_1, t)\} = \frac{1}{M_m}\{Y_m(x_1)\} \int_0^t h_m(t-\tau) \int_{-a/2}^{a/2} \{Z_m\}^T \{r_m(\xi, \tau)\} d\xi d\tau \tag{48}$$

where $h_m(t-\tau)$ is the impulse response function. To obtain the generalized displacements, we use Eqs. (18), (19), (20) and (48) in conjunction with Eq. (13).

Numerical Results and Discussion

The numerical applications are carried out for cross-ply spherical shallow shells whose geometrical and material properties are the same for all layers. The following material properties of a lamina in its principal coordinates are used:

$$E_2 = 1\times10^6 \text{ psi},\ E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,\ \nu_{12} = 0.25$$

The transverse deflection presented in the figures is evaluated at $(x_1, x_2, \zeta) = (0, b/2, \zeta)$. The stresses are nondimensionalized as follows:

$$\overline{\sigma_2} = \sigma_2(0, b/2, h/2)/q_0$$

$$\overline{\sigma_4} = \sigma_4(0, 0, 0)/q_0$$

In all calculations, unless otherwise stated, the following parameters are used (see Fig. 1)

$$a = b = 20 \text{ in},\ h = 2 \text{ in},\ R_1 = R_2 = 5a,$$

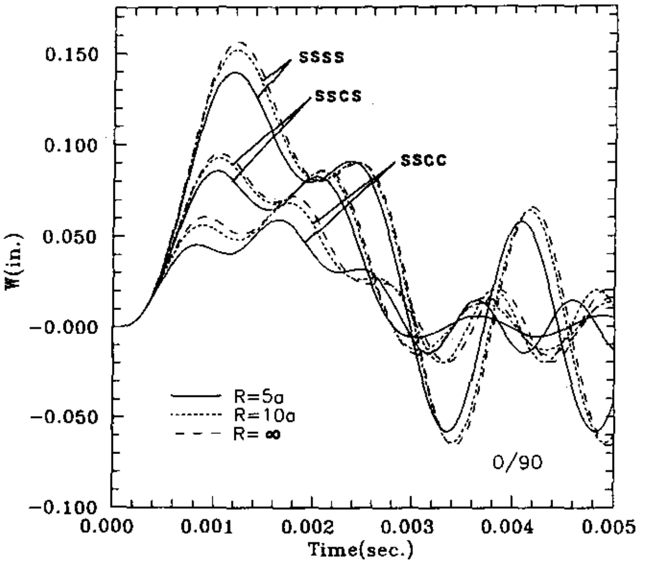
$$q_0 = 500 \text{ psi},\ t_1 = 0.003 \text{ sec},\ \rho = 0.00012 \text{ lb-s}^2/\text{in}^4$$

In all cases sinusoidal distribution of loading in spatial domain and sine pulse in time domain is used, $q(x_1, x_2, t) = q_0 \cos(\pi x_1/a)\sin(\pi x_2/b)F(t)$, where

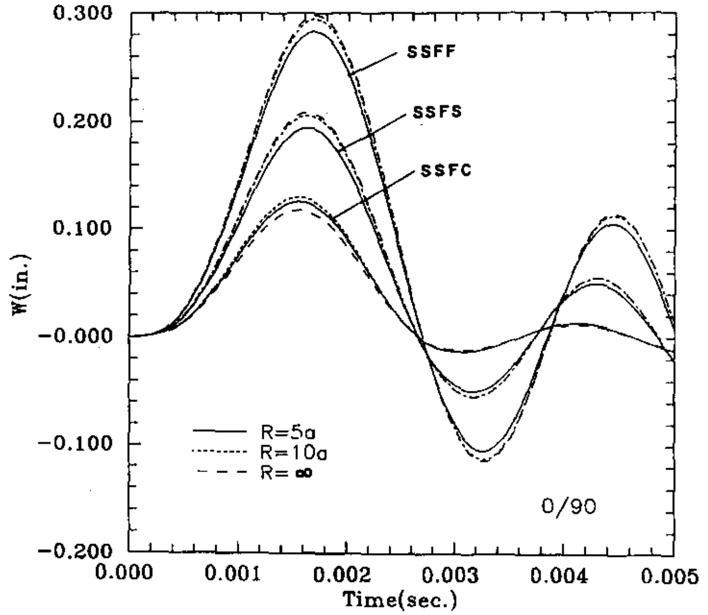
$$F(t) = \begin{cases} \sin(\frac{\pi t}{t_1}) & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

Zero initial conditions are assumed and for the first-order theory (FSDT), the shear correction coefficients are taken to be $K_{44}^2 = K_{55}^2 = 5/6$. For the explanation of S, C, and F in the figures, for example, SSFC means : the shell is simply supported (SS) at $x_2 = 0$ and $x_2 = b$, free (F) at $x_1 = a/2$ and clamped (C) at $x_1 = -a/2$.

The effect of shallowness of the shell on the center deflection of antisymmetric and symmetric cross-ply spherical caps are displaced in Fig. 4 and Fig. 5 respectively for various boundary conditions. All the results are obtained using the third-order theory. It is clear that the plate is relatively flexible when compared to the shell.



(a)



(b)

Fig. 4. Effect of shallowness of the shell on the center deflection of antisymmetric cross-ply spherical caps for various boundary conditions (a) SSSS, SSCS and SSCC. (b) SSFF, SSFS and SSFC.

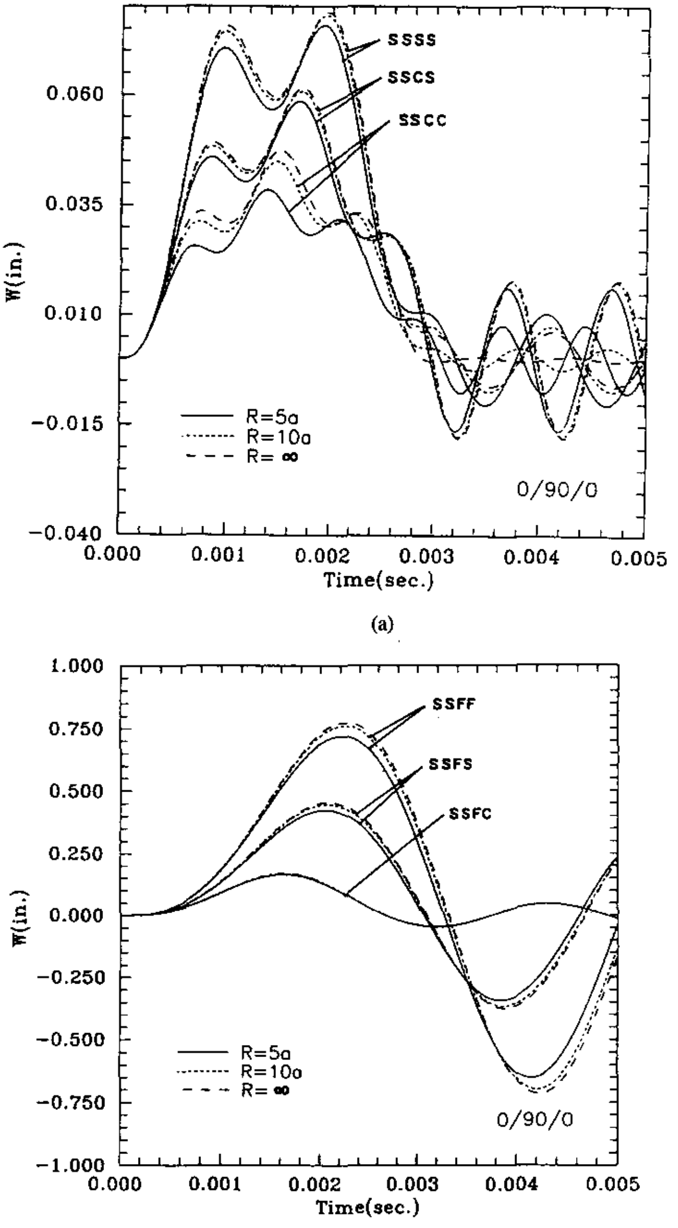
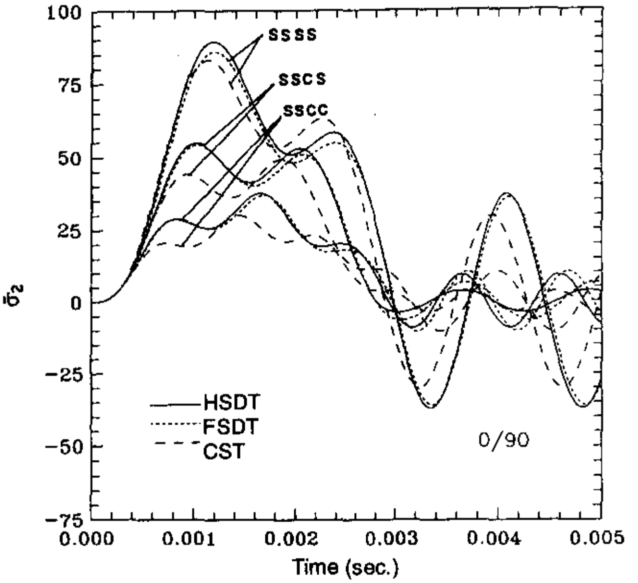
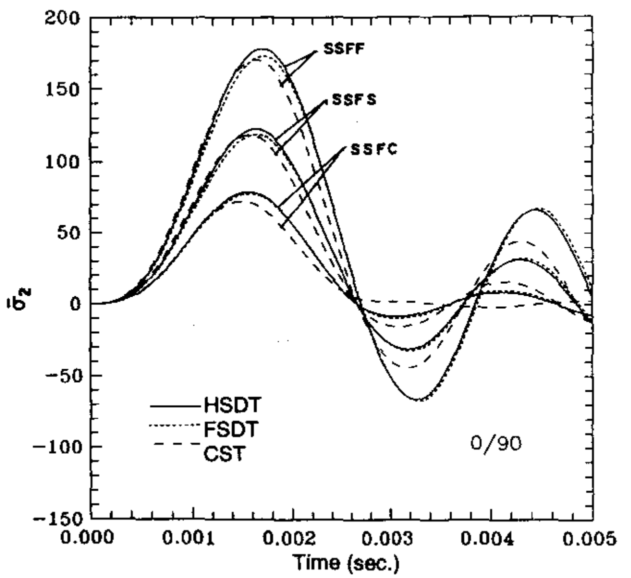


Fig. 5. Effect of shallowness of the shell on the center deflection of symmetric cross-ply spherical caps for various boundary conditions (a) SSSS, SSCS and SSCC. (b) SSFF, SSFS and SSFC.



(a)



(b)

Fig. 6. Normal stress ($\overline{\sigma_2}$) of two-layered spherical shells for various boundary conditions (a) SSSS, SSCS and SSCC. (b) SSFF, SSFS and SSFC.

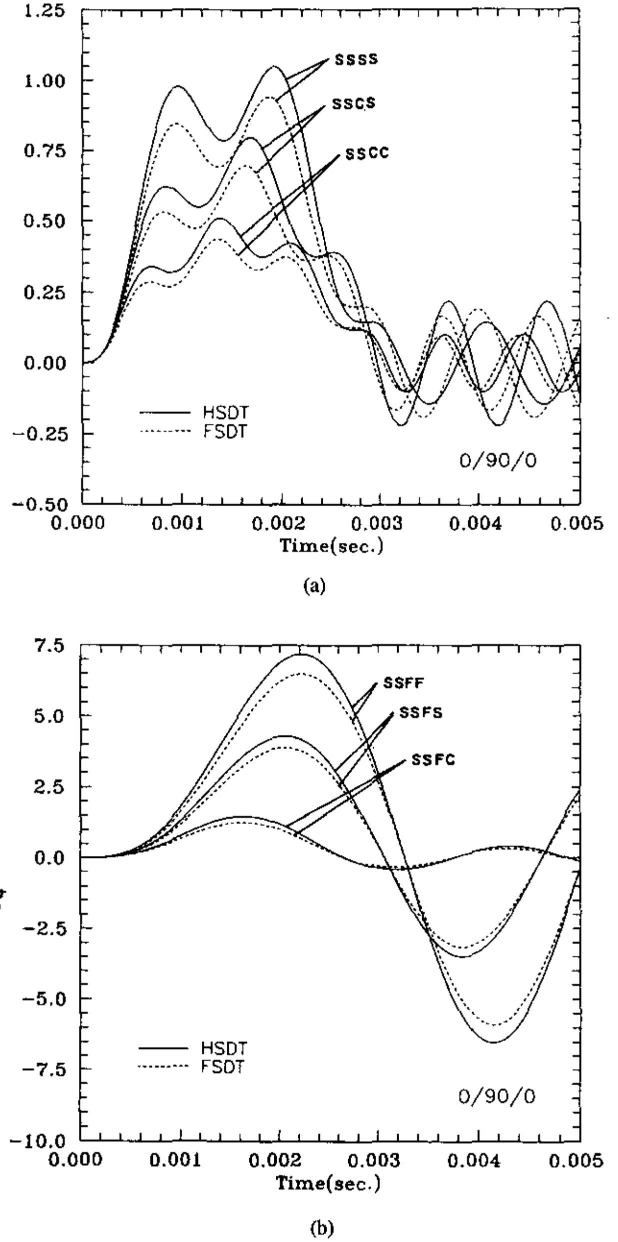


Fig. 7. Transverse shear stress ($\overline{\sigma_4}$) of three-layered spherical shells for various boundary conditions (a) SSSS, SSCS and SSCC. (b) SSFF, SSFS and SSFC.

The variation of the normal stress $\overline{\sigma}_2$ and the transverse shear stress $\overline{\sigma}_4$ with time are presented in Fig. 6 and Fig. 7 respectively. The normal stress $\overline{\sigma}_2$ obtained using the third-order and first-order theories are close and differ from the classical theory for all boundary conditions. Unlike the normal stress, the transverse shear stress $\overline{\sigma}_4$ predicted by the first-order theory differs significantly from that predicted by the third-order theory for all boundary conditions.

Conclusions

A generalized modal analysis approach is presented for forced vibration analysis of cross-ply laminated shallow shells. The equations of motion of the classical, first- and third-order theories are converted into a single-order system of equations by using state variables. The biorthogonality conditions of principal modes of the original and adjoint eigenfunctions are used to decouple the state space equations. An approach to utilize these modal quantities to obtain the forced response of shallow shells subjected to arbitrary loads is presented.

The numerical results for the deflection and stress responses are presented for shells with several different boundary conditions. To demonstrate the effect of shear deformation, the numerical results for the normal stress of antisymmetric cross-ply shells obtained by the third- and first- order theories are compared with the results obtained by the classical shell theory. It is noted that the first- and third- order theories seem to give very close results different than those of the classical theory because the classical theory ignores the effect of shear deformation. The normal stresses of shells with more end constraints (such as SSCC and SSCS shells) are seen to be more affected by the shear deformation than the shells with less end constraints (such as SSSS, SSFF, SSFS, SSFC) shells. As expected, difference is observed in the transverse shear stress calculated by the third-order and first-order theory because the third-order theory accounts for a layer-wise parabolic distribution of transverse shear stress whereas the first-order theory accounts for layer-wise constant states of transverse shear stress.

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Appendix A

The coefficients appearing in Eq. (15) are :

$c_1 = (e_7 e_{30} - e_3 e_{34})/e_0, c_2 = (e_2 e_{30} - e_3 e_{29})/e_0, c_3 = (e_6 e_{30} - e_3 e_{33})/e_0,$

$c_4 = (e_5 e_{30} - e_3 e_{32})/e_0, c_5 = (e_8 e_{30} - e_3 e_{35})/e_0, c_6 = (e_4 e_{30} - e_3 e_{31})/e_0,$

$c_7 = (e_9 e_{39} - e_{12} e_{36})/c_0, c_8 = (e_{14} e_{39} - e_{12} e_{41})/c_0, c_9 = (e_{16} e_{39} - e_{12} e_{43})/c_0,$

$c_{10} = (e_{13} e_{39} - e_{12} e_{40})/c_0, c_{11} = (e_{11} e_{39} - e_{12} e_{38})/c_0, c_{12} = (e_{15} e_{39} - e_{12} e_{42})/c_0,$

$c_{19} = (e_1 e_{34} - e_7 e_{28})/e_0, c_{20} = (e_1 e_{29} - e_2 e_{28})/e_0, c_{21} = (e_1 e_{33} - e_6 e_{28})/e_0,$

$c_{22} = (e_1 e_{32} - e_5 e_{28})/e_0, c_{23} = (e_1 e_{35} - e_8 e_{28})/e_0, c_{24} = (e_1 e_{31} - e_4 e_{28})/e_0,$

$c_{25} = (e_{10} e_{36} - e_9 e_{37})/c_0, c_{26} = (e_{10} e_{41} - e_{14} e_{37})/c_0, c_{27} = (e_{10} e_{43} - e_{16} e_{37})/c_0,$

$c_{28} = (e_{10} e_{40} - e_{13} e_{37})/c_0, c_{29} = (e_{10} e_{38} - e_{11} e_{37})/c_0, c_{30} = (e_{10} e_{42} - e_{15} e_{37})/c_0,$

$c_{13} = a_0(c_1 e_{21} + c_7 a_1 + c_{25} a_2 + c_{19} e_{23} + e_{26}), c_{14} = a_0(c_8 a_1 + c_{26} a_2 + e_{27}),$

$c_{15} = a_0(c_9 a_1 + c_{27} a_2 + e_{20}), c_{16} = a_0(c_{10} a_1 + c_{28} a_2 + c_3 e_{21} + c_{21} e_{23} + e_{18}),$

$c_{17} = a_0(c_{11} a_1 + c_{29} a_2 + c_5 e_{21} + c_{23} e_{23} + e_{17}), c_{18} = a_0(c_{12} a_1 + c_{30} a_2 + e_{19}),$

$b_1 = (e_3 m_{15} - e_{30} m_1)/e_0, b_2 = (e_3 m_{14} - e_{30} m_3)/e_0, b_3 = (e_3 m_{13} - e_{30} m_2)/e_0,$

$b_4 = (e_{12} m_{18} - e_{39} m_4)/c_0, b_5 = (e_{12} m_{17} - e_{39} m_6)/c_0, b_6 = (e_{12} m_{16} - e_{39} m_5)/c_0,$

$b_{13} = (e_{28} m_1 - e_1 m_{15})/e_0, b_{14} = (e_{28} m_3 - e_1 m_{14})/e_0, b_{15} = (e_{28} m_2 - e_1 m_{13})/e_0,$

$b_{16} = (e_{37} m_4 - e_{10} m_{18})/c_0, b_{17} = (e_{37} m_6 - e_{10} m_{17})/c_0, b_{18} = (e_{37} m_5 - e_{10} m_{16})/c_0,$

$b_7 = a_0(e_{21} b_1 + e_{23} b_{13} - m_{11}), b_8 = a_0(b_4 a_1 + b_{16} a_2 - m_{12}), b_9 = a_0(b_5 a_1 + b_{17} a_2 - m_7),$

$b_{10} = a_0(e_{21} b_2 + e_{23} b_{14} - m_8), b_{11} = a_0(e_{21} b_3 + e_{23} b_{15} - m_9),$

$b_{12} = a_0(b_6 a_1 + b_{18} a_2 - m_{10}), e_0 = e_3 e_{28} - e_1 e_{30}, c_0 = e_{12} e_{37} - e_{10} e_{39},$

$a_0 = -1/(c_4 e_{21} + c_{22} e_{23} + e_{25}), a_1 = c_2 e_{21} + c_{20} e_{23} + e_{22}, a_2 = c_6 e_{21} + c_{24} e_{23} + e_{24},$

where

$$e_1 = A_{11}, e_2 = -\beta(A_{12}+A_{66}), e_3 = B_{11}-n_2E_{11},$$
$$e_4 = \beta[n_2(E_{12}+E_{66})-B_{12}-B_{66}], e_5 = -n_2E_{11},$$
$$e_6 = n_2[\beta^2(E_{12}+2E_{66})]+A_{11}/R_1+A_{12}/R_2, e_7 = -\beta^2A_{66},$$
$$e_8 = \beta^2(n_2E_{66}-B_{66}), e_9 = -e_2, e_{10} = A_{66},$$
$$e_{11} = -e_4, e_{12} = B_{66}-n_2E_{66}, e_{13} = -\beta n_2(E_{12}+2E_{66}), e_{14} = -\beta^2A_{22},$$
$$e_{15} = \beta^2(n_2E_{22}-B_{22}), e_{16} = n_2\beta^3E_{22}+\beta(A_{12}/R_1+A_{22}/R_2),$$
$$e_{17} = A_{55}-n_1D_{55}-n_1(D_{55}-n_1F_{55})+$$
$$n_2\beta^2[n_2(H_{12}+2H_{66})-(F_{12}+2F_{66})]-(B_{11}-n_2E_{11})/R_1-(B_{12}-n_2E_{12})/R_2,$$
$$e_{18} = A_{55}-n_1D_{55}-n_1(D_{55}-n_1F_{55})+n_2^2\beta^2[2H_{12}+4H_{66}]$$
$$+2n_2(E_{11}/R_1+E_{12}/R_2),$$
$$e_{19} = -\beta[A_{44}-n_1D_{44}-n_1(D_{44}-n_1F_{44})]+n_2\beta^3(F_{22}-n_2H_{22})$$
$$+\beta(B_{12}-n_2E_{12})/R_1+\beta(B_{22}-n_2E_{22})/R_2,$$
$$e_{20} = -\beta^2[A_{44}-n_1D_{44}-n_1(D_{44}-n_1F_{44})]-n_2^2\beta^4H_{22}$$
$$-2n_2\beta^2(E_{12}/R_1+E_{22}/R_2)-A_{11}/R_1^2-2A_{12}/(R_1R_2)-A_{22}/R_2^2,$$
$$e_{21} = -e_5, e_{22} = e_{13}, e_{23} = n_2(F_{11}-n_2H_{11}),$$
$$e_{24} = \beta n_2[n_2(2H_{66}+H_{12})-(F_{12}+2F_{66})], e_{25} = -n_2^2H_{11}, e_{26} = -e_6,$$
$$e_{27} = e_{16}, e_{28} = e_3, e_{29} = e_4, e_{30} = D_{11}-2n_2F_{11}+n_2^2H_{11},$$
$$e_{31} = \beta[2n_2(F_{12}+F_{66})-n_2^2(H_{12}+H_{66})-D_{12}-D_{66}], e_{32} = -e_{23}, e_{33} = -e_{17},$$
$$e_{34} = e_8, e_{35} = n_1(D_{55}-n_1F_{55})-(A_{55}-n_1D_{55})+\beta^2[2n_2F_{66}-D_{66}-n_2^2H_{66}],$$
$$e_{36} = -e_4, e_{37}=e_{12}, e_{38} = -e_{31}, e_{39} = D_{66}-2n_2F_{66}+n_2^2H_{66}, e_{40} = e_{24}, e_{41} = e_{15},$$
$$e_{42} = n_1(D_{44}-n_1F_{44})-(A_{44}-n_1D_{44})+\beta^2[2n_2F_{22}-D_{22}-n_2^2H_{22}], e_{43} = e_{19},$$
$$m_1 = \overline{I_1}, m_2 = \overline{I_2}, m_3 = -\overline{I_3}, m_4 = \overline{I_1}', m_5 = \overline{I_2}', m_6 = -\beta\overline{I_3}', m_7 = I_1+\beta^2n_2^2I_7,$$
$$m_8 = -n_2^2I_7, m_9 = \overline{I_5}, m_{10} = -\beta\overline{I_5}, m_{11} = -m_3, m_{12} = m_6, m_{13} = \overline{I_4}, m_{14} = -m_9,$$
$$m_{15} = m_2, m_{16} = m_{13}, m_{17} = m_{10}, m_{18} = m_5,$$

The coefficients appearing in Eq. (16) are:

$$c_1 = (e_3 e_{21} - e_5 e_{19})/e_0, c_2 = (e_3 e_{18} - e_2 e_{19})/e_0, c_3 = (e_3 e_{23} - e_{19} e_{31})/e_0,$$
$$c_4 = (e_3 e_{22} - e_6 e_{19})/e_0, c_5 = (e_3 e_{20} - e_4 e_{19})/e_0, c_6 = (e_8 e_{27} - e_{10} e_{25})/c_0,$$
$$c_7 = (e_{11} e_{27} - e_{10} e_{28})/c_0, c_8 = (e_{27} e_{32} - e_{10} e_{30})/c_0, c_9 = (e_9 e_{27} - e_{10} e_{26})/c_0,$$
$$c_{10} = (e_{12} e_{27} - e_{10} e_{29})/c_0, c_{11} = -e_{33}/e_{13}, c_{12} = -e_{34}/e_{13}, c_{13} = -e_{15}/e_{13},$$
$$c_{14} = -e_{14}/e_{13}, c_{15} = -e_{16}/e_{13}, c_{16} = (e_5 e_{17} - e_1 e_{21})/e_0, c_{17} = (e_2 e_{17} - e_1 e_{18})/e_0,$$
$$c_{18} = (e_{17} e_{31} - e_1 e_{23})/e_0, c_{19} = (e_6 e_{17} - e_1 e_{22})/e_0, c_{20} = (e_4 e_{17} - e_1 e_{20})/e_0,$$
$$c_{21} = (e_7 e_{25} - e_8 e_{24})/c_0, c_{22} = (e_7 e_{28} - e_{11} e_{24})/c_0, c_{23} = (e_7 e_{30} - e_{24} e_{32})/c_0,$$
$$c_{24} = (e_7 e_{26} - e_9 e_{24})/c_0, c_{25} = (e_7 e_{29} - e_{12} e_{24})/c_0, b_1 = (e_{19} I_1 - e_3 I_2)/e_0,$$
$$b_2 = (e_{19} I_2 - e_3 I_3)/e_0, b_3 = (e_{10} I_2 - e_{27} I_1)/c_0, b_4 = (e_{10} I_3 - e_{27} I_2)/c_0,$$
$$b_5 = I_1/e_{13}, b_6 = (e_1 I_2 - e_{17} I_1)/e_0, b_7 = (e_1 I_3 - e_{17} I_2)/e_0, b_8 = (e_{24} I_1 - e_7 I_2)/c_0,$$
$$b_9 = (e_{24} I_2 - e_7 I_3)/c_0, e_0 = e_1 e_{19} - e_3 e_{17}, c_0 = e_{10} e_{24} - e_7 e_{27}, a_0 = -1/e_{13}.$$

where

$$e_1 = A_{11}, e_2 = -\beta(A_{12} + A_{66}), e_3 = B_{11}, e_4 = -\beta(B_{12} + B_{66}), e_5 = -\beta^2 A_{66},$$
$$e_6 = -\beta^2 B_{66}, e_7 = A_{66}, e_8 = -e_2, e_9 = -e_4, e_{10} = B_{66}, e_{11} = -\beta^2 A_{22},$$
$$e_{12} = -\beta^2 B_{22}, e_{13} = K_{55}^2 A_{55}, e_{14} = K_{55}^2 A_{55} - B_{11}/R_1 - B_{12}/R_2,$$
$$e_{15} = -\beta^2 K_{44}^2 A_{44} - A_{11}/R_1^2 - 2 A_{12}/(R_1 R_2) - A_{22}/R_2^2,$$
$$e_{16} = -\beta K_{44}^2 A_{44} + \beta(B_{12}/R_1 + B_{22}/R_2), e_{17} = e_3, e_{18} = e_4,$$
$$e_{19} = D_{11}, e_{20} = -\beta(D_{12} + D_{66}), e_{21} = e_6, e_{22} = -\beta^2 D_{66} - K_{55}^2 A_{55},$$
$$e_{23} = -e_{14}, e_{24} = e_{10}, e_{25} = -e_4, e_{26} = -e_{20}, e_{27} = D_{66},$$
$$e_{28} = e_{12}, e_{29} = -\beta^2 D_{22} - K_{44}^2 A_{44}, e_{30} = e_{16}, e_{31} = A_{11}/R_1 + A_{12}/R_2,$$
$$e_{32} = \beta(A_{12}/R_1 + A_{22}/R_2), e_{33} = -e_{31}, e_{34} = e_{32}.$$

The coefficients appearing in Eq. (17) are:

$$c_1 = -e_2/e_1, c_2 = -e_3/e_1, c_3 = -e_5/e_1, c_4 = -e_4/e_1, c_5 = -e_6/e_7,$$
$$c_6 = -e_8/e_7, c_7 = -e_{10}/e_7, c_8 = -e_9/e_7, c_9 = -e_{21}/e_{18}, c_{10} = -e_{22}/e_{18},$$
$$c_{11} = -e_{20}/e_{18}, c_{12} = -e_{19}/e_{18}, b_1 = I_1/e_1, b_2 = -I_2/e_1,$$
$$b_3 = I_1/e_7, b_4 = -\beta I_2/e_7, b_5 = -I_1 e_{14}/(e_1 e_{18}) - I_2/e_{18},$$
$$b_6 = -I_1 e_{16}/(e_7 e_{18}) + I_1 e_3 e_{14}/(e_1 e_7 e_{18}) + \beta I_2/e_{18},$$
$$b_7 = -(I_1 + \beta^2 I_3)/e_{18} + \beta I_2 e_{16}/(e_7 e_{18}) - \beta I_2 e_3 e_{14}/(e_1 e_7 e_{18}),$$
$$b_8 = I_3/e_{18} + I_2 e_{14}/(e_1 e_{18}), a_0 = 1/e_{18}.$$

where

$$e_1 = A_{11}, e_2 = -\beta^2 A_{66}, e_3 = -\beta(A_{12} + A_{66}), e_4 = -B_{11},$$
$$e_5 = \beta^2(B_{12} + 2B_{66}) + A_{11}/R_1 + A_{12}/R_2, e_6 = -e_3, e_7 = A_{66}, e_8 = -\beta^2 A_{22},$$
$$e_9 = -\beta(B_{12} + 2B_{66}), e_{10} = \beta^3 B_{22} + \beta(A_{12}/R_1 + A_{22}/R_2), e_{11} = D_{11},$$
$$e_{12} = -2\beta^2(D_{12} + 2D_{66}) - 2(B_{11}/R_1 + B_{12}/R_2),$$
$$e_{13} = \beta^4 D_{22} + A_{11}/R_1^2 + 2A_{12}/(R_1 R_2) + A_{22}/R_2^2 + 2\beta^2(B_{12}/R_1 + B_{22}/R_2),$$
$$e_{14} = e_4, e_{15} = e_5, e_{16} = -e_9, e_{17} = -e_{10}, e_{18} = e_{11} - e_4 e_{14}/e_1,$$
$$e_{19} = e_{12} - e_5 e_{14}/e_1 - e_9 e_{16}/e_7 + e_3 e_9 e_{14}/(e_1 e_7),$$
$$e_{20} = e_{13} - e_{10} e_{16}/e_7 + e_3 e_{10} e_{14}/(e_1 e_7),$$
$$e_{21} = e_{15} - e_2 e_{14}/e_1 - e_6 e_{16}/e_7 + e_3 e_6 e_{14}/(e_1 e_7),$$
$$e_{22} = e_{17} - e_8 e_{16}/e_7 + e_3 e_8 e_{14}/(e_1 e_7).$$

Appendix B

The matrices [K] and [M] in Eq. (21)

HSDT

$$[\,K\,]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ c_{15} & 0 & c_{16} & 0 & c_{14} & 0 & c_{18} & 0 & c_{13} & 0 & c_{17} & 0 \\ 0 & c_3 & 0 & c_1 & 0 & c_5 & 0 & c_4 & 0 & c_2 & 0 & c_6 \\ c_9 & 0 & c_{10} & 0 & c_8 & 0 & c_{12} & 0 & c_7 & 0 & c_{11} & 0 \\ 0 & c_{21} & 0 & c_{19} & 0 & c_{23} & 0 & c_{22} & 0 & c_{20} & 0 & c_{24} \\ c_{27} & 0 & c_{28} & 0 & c_{26} & 0 & c_{30} & 0 & c_{25} & 0 & c_{29} & 0 \end{bmatrix}$$

$$
[\ M\ ]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_9 & 0 & b_{10} & 0 & b_8 & 0 & b_{12} & 0 & b_7 & 0 & b_{11} & 0 \\
0 & b_2 & 0 & b_1 & 0 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_5 & 0 & 0 & 0 & b_4 & 0 & b_6 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{14} & 0 & b_{13} & 0 & b_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
b_{17} & 0 & 0 & 0 & b_{16} & 0 & b_{18} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

FSDT

$$
[\ K\ ]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
c_{13} & 0 & c_{12} & 0 & c_{15} & 0 & c_{11} & 0 & c_{14} & 0 \\
0 & c_1 & 0 & c_4 & 0 & c_3 & 0 & c_2 & 0 & c_5 \\
c_8 & 0 & c_7 & 0 & c_{10} & 0 & c_6 & 0 & c_9 & 0 \\
0 & c_{16} & 0 & c_{19} & 0 & c_{18} & 0 & c_{17} & 0 & c_{20} \\
c_{23} & 0 & c_{22} & 0 & c_{25} & 0 & c_{21} & 0 & c_{24} & 0
\end{bmatrix}
$$

$$[\ M\ ]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & b_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_6 & 0 & b_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_8 & 0 & b_9 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

CST

$$[\ K\ ]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ c_{11} & 0 & c_{12} & 0 & c_{10} & 0 & c_9 & 0 \\ 0 & c_3 & 0 & c_1 & 0 & c_4 & 0 & c_2 \\ c_7 & 0 & c_8 & 0 & c_6 & 0 & c_5 & 0 \end{bmatrix}$$

$$
[\,M\,] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_7 & 0 & b_8 & 0 & b_6 & 0 & b_5 & 0 \\
0 & b_2 & 0 & b_1 & 0 & 0 & 0 & 0 \\
b_4 & 0 & 0 & 0 & b_3 & 0 & 0 & 0
\end{bmatrix}
$$

Appendix C

The coefficients d_i appearing in Eq. (39) are:

$$
\begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{bmatrix} = -\begin{bmatrix} s_3 & s_4 & s_2 \\ s_9 & s_{10} & s_8 \\ s_{15} & s_{16} & s_{14} \end{bmatrix}^{-1} \begin{bmatrix} s_6 & s_1 & s_5 \\ s_{12} & s_7 & s_{11} \\ s_{18} & s_{13} & s_{17} \end{bmatrix}
$$

$$
\begin{bmatrix} d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} \\ d_{16} & d_{17} & d_{18} \end{bmatrix} = -\begin{bmatrix} s_{21} & s_{19} & s_{22} \\ s_{26} & s_{24} & s_{27} \\ s_{39} & s_{37} & s_{40} \end{bmatrix}^{-1} \begin{bmatrix} 0 & s_{20} & s_{23} \\ 0 & s_{25} & s_{28} \\ s_{42} & s_{38} & s_{41} \end{bmatrix}
$$

where

$$s_1 = A_{11}, s_2 = -\beta A_{12}, s_3 = n_2\beta^2 E_{12} + A_{11}/R_1 + A_{12}/R_2, s_4 = -n_2 E_{11},$$
$$s_5 = B_{11} - n_2 E_{11}, s_6 = \beta(n_2 E_{12} - B_{12}), s_7 = B_{11}, s_8 = -\beta B_{12},$$
$$s_9 = \beta^2 n_2 F_{12} + B_{11}/R_1 + B_{12}/R_2, s_{10} = -n_2 F_{11},$$
$$s_{11} = D_{11} - n_2 F_{11}, s_{12} = \beta(n_2 F_{12} - D_{12}), s_{13} = E_{11}, s_{14} = -\beta E_{12},$$
$$s_{15} = n_2\beta^2 H_{12} + E_{11}/R_1 + E_{12}/R_2, s_{16} = -n_2 H_{11}, s_{17} = F_{11} - n_2 H_{11},$$
$$s_{18} = \beta(n_2 H_{12} - F_{12}), s_{19} = \beta A_{66}, s_{20} = A_{66}, s_{21} = -2\beta n_2 E_{66},$$
$$s_{22} = \beta(B_{66} - n_2 E_{66}), s_{23} = B_{66} - n_2 E_{66}, s_{24} = s_{22}, s_{25} = s_{23},$$
$$s_{26} = 2\beta n_2(n_2 H_{66} - F_{66}), s_{27} = \beta(D_{66} - 2n_2 F_{66} + n_2^2 H_{66}),$$
$$s_{28} = s_{27}/\beta, s_{29} = A_{55} - n_1 D_{55} - n_1(D_{55} - n_1 F_{55}) + \beta^2 n_2(n_2 H_{66} - F_{66}),$$
$$s_{30} = A_{55} - n_1 D_{55} - n_1(D_{55} - n_1 F_{55}) + \beta^2 n_2^2(H_{12} + 2H_{66}) + n_2(E_{11}/R_1 + E_{12}/R_2),$$
$$s_{31} = n_2 E_{11}, s_{32} = -\beta n_2(E_{12} + E_{66}), s_{33} = n_2(F_{11} - n_2 H_{11}),$$
$$s_{34} = \beta n_2(n_2 H_{12} + n_2 H_{66} - F_{12} - F_{66}), s_{35} = -n_2^2 H_{11}, s_{36} = -n_2\beta^2 E_{66},$$
$$s_{37} = s_{31}(c_1 - \omega_m^2 b_1) + s_{33}(c_{19} - \omega_m^2 b_{13}) + s_{36}, s_{38} = s_{31}c_2 + s_{33}c_{20} + s_{32},$$
$$s_{39} = s_{30} + s_{31}(c_3 - \omega_m^2 b_2) + s_{33}(c_{21} - \omega_m^2 b_{14}),$$
$$s_{40} = s_{29} + s_{31}(c_5 - \omega_m^2 b_3) + s_{33}(c_{23} - \omega_m^2 b_{15}),$$
$$s_{41} = s_{31}c_6 + s_{33}c_{24} + s_{34}, s_{42} = s_{31}c_4 + s_{33}c_{22} + s_{35}.$$

The coefficients d_i appearing in Eq. (40) are:

$$d_1 = [B_{11}(B_{11}/R_1 + B_{12}/R_2) - D_{11}(A_{11}/R_1 + A_{12}/R_2)]/e_0,$$
$$d_2 = \beta(A_{12}D_{11} - B_{11}B_{12})/e_0,$$
$$d_3 = \beta(B_{12}D_{11} - B_{11}D_{12})/e_0, d_4 = (A_{12}B_{11} - A_{11}B_{12})/(e_0 R_2),$$
$$d_5 = \beta(A_{11}B_{12} - A_{12}B_{11})/e_0, d_6 = \beta(A_{11}D_{12} - B_{11}B_{12})/e_0.$$

The coefficients d_i appearing in Eq. (41) are:

$$d_1 = (s_4 s_6 - s_2 s_8)/s_0, d_2 = (s_4 s_5 - s_1 s_8)/s_0, d_3 = (s_2 s_7 - s_3 s_6)/s_0,$$
$$d_4 = (s_1 s_7 - s_3 s_5)/s_0, d_5 = s_9 s_{15}/d_0, d_6 = (s_9 s_{13} - s_{10} s_{12})/d_0,$$
$$d_7 = -s_{11} s_{15}/d_0, d_8 = (s_{10} s_{14} - s_{11} s_{13})/d_0,$$
$$s_0 = s_3 s_8 - s_4 s_7, d_0 = s_{11} s_{12} - s_9 s_{14},$$

where

$$s_1 = A_{11}, s_2 = -\beta A_{12}, s_3 = \beta^2 B_{12} + A_{11}/R_1 + A_{12}/R_2, s_4 = -B_{11}, s_5 = B_{11},$$
$$s_6 = -\beta B_{12}, s_7 = \beta^2 D_{12} + B_{11}/R_1 + B_{12}/R_2, s_8 = -D_{11}, s_9 = \beta(A_{66} + B_{66}/R_2),$$
$$s_{10} = s_9/\beta, s_{11} = -2\beta(B_{66} + D_{66}/R_2),$$
$$s_{12} = -B_{11} e_2/e_1 - \omega_m^2 I_1 B_{11}/e_1 - 2\beta^2 B_{66}, s_{13} = -B_{11} e_3/e_1 - \beta B_{12} - 2\beta B_{66},$$
$$s_{14} = \beta^2 (D_{12} + 4 D_{66}) + B_{11}/R_1 + B_{12}/R_2 - B_{11} e_5/e_1 + \omega_m^2 B_{11} I_2/e_1,$$
$$s_{15} = -B_{11} e_4/e_1 - D_{11}.$$

تحليل الاستجابة الديناميكية للقشريات الضحلة المكونة من طبقات متعددة الألياف لمختلف شروط الحافة

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ملخص البحث. تم تطوير حلول تحليلية للاستجابة الديناميكية للنظريات التقليدية ذات الدرجة الأولى وذات الدرجة الثالثة للقشريات الضحلة المكونة من طبقات متعددة الألياف ولشروط حافة مختلفة. تنطبق هذه الحلول على قشريات طبقية بحافتين متقابلتين ذات ركائز بسيطة بينما بقية الحواف هي مزيج اختياري من شروط الحافة بين الركائز البسيطة أو الحرة أو المثبتة. استخدمت طريقة ليفي بالاقتران مع الاتجاه العام الشكلي للحصول على هذه الحلول. وفي حالة القشريات السميكة، فإن النتائج التي حصلنا عليها من نظرية القشرة التقليدية للإجهادات والانحرافات تختلف بشكل بارز عن تلك التي حصلنا عليها في حالة استخدام نظرية الدرجة الثالثة. كما أن النتائج التي حصلنا عليها من نظرية الدرجة الثالثة ونظرية الدرجة الأولى متقاربة في حالة الانحرافات والإجهادات المتعامدة. ومن ناحية أخرى، إن نظرية الدرجة الثالثة لا تتطلب استخدام معاملات تصحيحية قصية.