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# BRIEF ARTICLE

## Simplified Model for Hollow Fine Fiber Modules with Radial Dispersion

**Ibrahim S. Al-Mutaz, M.A. Soliman and A.E.S. Al-Zahrani**
*Chemical Engineering Department, College of Engineering*
*King Saud University, P.O. Box 800, Riyadh, Saudi Arabia*

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**Abstract.** There is enough experimental evidence indicating that the reciprocal of the rejection of hollow fine fiber modules is linearly related to the reciprocal of the product flow rate. We show that this relationship can be obtained from a rigorous mathematical model that accounts for radial dispersion using approximations based on the application of one point collocation method.

### List of Symbols

| | |
|---|---|
| A | permeability of membrane, $cm^2$ sec/gm. |
| $c_f$ | feed concentration, gm solute/gm solvent. |
| $c_1$ | shell side concentration, gm solute/gm solvent. |
| $C_1$ | dimensionless shell side concentration = $C_r/C_f$. |
| $C_{1,1}$ | dimensionless shell side concentration at one point collocation. |
| $C_{1,i}$ | shell side concentration at the inlet point, gm solute/gm solvent. |
| $c_{1,i}$ | dimensionless shell side concentration at the inlet point. |
| $c_{1,0}$ | outlet concentration from the shell, gm solute/gm solvent. |
| $C_{1,o}$ | dimensionless shell side outlet concentration. |
| $c_3$ | fiber side concentration, gm solute/gm solvent. |
| $C_3$ | dimensionless fiber side concentration = $c_3/c_f$. |
| $C_3$ | dimensionless fiber side concentration averaged over the fiber length. |

$c_p$ product concentration, gm solute/gm solvent.

$C_p$ dimensionless product concentration = $c_p/c_f$.

$\overline{C}_P$ product concentration, averaged over the fiber length.

$c_R$ reject concentration, gm solute/gm solvent.

$C_R$ dimensionless reject concentration = $C_R/C_F$.

D diffusivity, $cm^2$/sec.

$F_F$ feed flowrate, $cm^3$/sec.

$F_P$ product flowrate, $cm^3$/sec.

K diffusion constant, cm/sec.

$K_o$ fiber orientation constant.

l active length of fiber, cm.

$l_s$ seal length, cm.

$L_s$ dimensionless seal length = $l_s/l$.

N dimensionless brine flux.

$N_1$ dimensionless brine flux at one point collocation.

$N_i$ dimensionless brine flux at the inlet point.

$p_1$ shell side pressure, gm/cm $sec^2$.

$p_f$ feed pressure, gm/cm $sec^2$.

$P_1$ dimensionless shell side pressure = $p_1/(p_f - p_{atm})$.

$P_{1,1}$ dimensionless shell side pressure at one point collocation.

$P_f$ dimensionless feed pressure = $p_f/(p_f - p_{atm})$.

$p_3$ fiber side pressure, gm/cm $sec^2$.

$P_3$ dimensionless fiber side pressure = $p_3/(p_f - p_{atm})$.

$P_{3,1}$ dimensionless fiber side pressure at one point collocation.

$p_{atm}$ atmospheric pressure, gm/sec $sec^2$.

$P_{atm}$ dimensionless atmospheric pressure = $p_{atm}/(p_f - p_{atm})$.

$p_{3E}$ exit pressure from fiber, gm/cm $sec^2$.

$P_{3E}$ dimensionless exit pressure from fiber.

$P_e$ Peclet number = $\dfrac{v_f\, r_i}{D}$.

q local permeation rate into fibers/unit volume, $sec^{-1}$.

$\overline{Q}$ Dimensionless permeation rate = $\dfrac{\overline{q}\,\overline{r}_l}{v_f}$.

$\bar{r}$ radial coordinate, cm

$r$ dimensionless coordinate defined as $[r = (R-1)/(R_o-1)]$

$r_i$ inside fiber radius, cm

$r_o$ outside fiber radius, cm

$\bar{r}_i$ central feeder radius, cm

$\bar{r}_o$ fiber bundle radius, cm

$R$ dimensionless radial coordinate $= \frac{\bar{r}}{\bar{r}_i}$.

$R_o$ dimensionless fiber bundle radius $= \frac{\bar{r}_o}{\bar{r}_i}$.

$s_o$ specific surface area of fibers, $cm^2/cm^3$ ($s_o = 2/r_o$).

$S_m$ effective surface area of the membranes, $cm^2$.

$v_f$ radial feed velocity, cm/sec.

$v_1$ shell side velocity, cm/sec.

$V_1$ dimensionless shell side velocity $= \frac{v_1}{v_f}$

$V_{1,1}$ dimensionless shell side velocity at one point collocation.

$v_w$ wall permeation velocity for pure water, cm/sec $= -A\,(p_F - p_{atm})$.

$V_w$ dimensionless wall permeation velocity $= \frac{v_w}{v_w^*}$.

$\overline{V}_w$ wall permeation velocity, on the average length, cm/sec.

$z$ axial coordinate, cm.

$Z$ dimensionless axial coordinate $= z/1$.

**Greek symbols**

$\beta$ dimensionless quantity defined by equation (17).

$\gamma$ dimensionless quantity $= (\pi f) / (p_F - p_{atm})$.

$\Theta$ dimensionless quantity $= \frac{-K}{v_w^*}$.

$\phi_1$ dimensionless quantity $= \dfrac{2(1-\varepsilon)\upsilon_w^* \bar{r}_i}{r_o \upsilon_f}$.

$\phi_2$ dimensionless quantity $= \dfrac{2 r_o L}{r_i^2 \upsilon_f} \upsilon_w^*$.

$\phi_3$ dimensionless quantity $= \dfrac{8 \mu L \upsilon_f}{r_i^2 (p_f - p_{atm})}$.

$\phi_4$ dimensionless quantity $= \dfrac{K_o \mu S_o \dfrac{(1-\varepsilon)^2}{\varepsilon^2} \dfrac{V_f}{\varepsilon} \bar{r}_i}{(p_f - p_{atm})}$.

$\varepsilon$ porosity of fiber bundle.

$\Delta$ productivity = $F_p/F_F$.

$\pi f$ feed osmotic pressure, gm/cm sec$^2$.

$\mu 1$ feed viscosity, gm/c, sec.

## Introduction

Many workers have presented rigorous as well as simple models to describe the performance of hollow fine fiber modules. Dandavati *et al.* [1] presented a mathematical model that forms the basis for most of the subsequent work. This model is based on the assumption that the radial flow of the shell side liquid is of the plug flow type, i.e., there is no back mixing. This model, however, was not able to account for the observation that the reciprocal of the rejection is related linearly to the reciprocal of the product flow rate. Soltanieh and Gill [2] used the assumption of complete mixing in the shell side to account for this observation. Gill *et al.* [3] presented experimental evidence based on effluent tracer concentration distribution that the flow is in fact partially mixed.

Mehdizadeh and Dickson [4] solved the membrane transport differential equations using orthogonal collocation. Malek *et al.* [5] presented a lumped parameter model which is basically a complete mixing model based on the assumption that the salt concentration in the shell is the arithmetic average of inlet and outlet concentration.

The authors, in a previous paper [6], used a one point collocation method to simplify the partial differential equations representing the steady state behavior of hollow fine fiber modules and discretize them into a system of algebraic equations. The equations were not simple enough. The purpose of this paper is to further simplify the model equations, to manipulate the simplified equations to show the linear relationship between the reciprocal of the rejection and the reciprocal of the product flow rate and to estimate approximately the Peclet number.

## Development of the Approximate Equations

In reference [6], the describing equations for hollow fine fiber modules are given as the dimensionless equations describing:

(1) The radial convective diffusion of salt in the shell

$$\frac{1}{(R_o-1)}\frac{dN}{dr} - Pe\,NV_1 + (C_3 - C_1)\overline{Q}(1+(R_o-1)r) = 0 \tag{1}$$

$$\frac{(1+(R_o-1)r)}{Pe(R_o-1)}\frac{dC_1}{dr} = N \tag{2}$$

(2) The radial water balance in the shell

$$\frac{1}{(R_o-1)}\frac{d[(1+((R_o-1)r)V_1)]}{dr} = (1+(R_o-1)r)\overline{Q} \tag{3}$$

(3) The radial pressure drop in the shell

$$\frac{1}{(R_o-1)}\frac{dP_1}{dr} = -\phi_4 V_1 \tag{4}$$

(4) The permeation rate through the fibers.

$$\overline{Q} = \phi_1 \overline{V}_w \tag{5}$$

(5) The wall permeation velocity averaged on the fiber length

$$\overline{V_w} = \int_o^1 V_w \, dZ \tag{6}$$

(6) The axial pressure drop inside the fiber

$$\frac{d^2 P_3}{dZ^2} = -\phi_2\phi_3 V_w \tag{7}$$

(7) The axial exit pressure from the fiber

$$P_{3E} = P_{atm} - \Phi_2\Phi_3 L_S \overline{V}_w \tag{8}$$

(8) The wall permeation velocity

$$V_w - P_1 - P_3 - \gamma(C_1 - C_3) \tag{9}$$

(9) The diffusion of salt across the fiber

$$C_3 V_w = \theta (C_1 - C_3) \tag{10}$$

with the boundary conditions:

| | | | |
|---|---|---|---|
| $N = C_1 - 1$ | at | $r = 0$ | (11) |
| $N = 0$ | at | $r = 1$ | (12) |
| $V_1 = 1$ | at | $r = 0$ | (13) |
| $P_1 = P_{1f}$ | at | $r = 0$ | (14) |
| $dP_3/dZ = 0$ | at | $Z = 0$ | (15) |
| $P_3 = P_{3E}$ | at | $Z = 1$ | (16) |

The one point collocation method is suitable for approximating boundary value problems when the dependent variable can be approximated by a quadratic in the independent variable. In our case this is valid for large Peclet number and small pressure drops in the radial and axial directions.

The one point collocation method is applied at the dimensionless longitudinal distance $Z = \sqrt{0.2}$ and the dimensionless radial distance r = 1/2. This application led to the following simple equations [6]:

(I) for the average permeate Vw,

$$\overline{V}_w - \beta (P_f - P_{atm} - \gamma \overline{R} C_{1,1}) \tag{17}$$

where,

$$\beta = \frac{\dfrac{1 - \Phi_2 \Phi_3}{15}}{1 - 0.4 \Phi_2 \Phi_3 - \Phi_2 \Phi_3 L_s \left( \dfrac{1 - \Phi_2 \Phi_3}{15} \right)}$$

and

$$\overline{R} = 1 - \frac{C_{3,1}}{C_{1,1}}$$

where $C_{3,1}$ and $C_{1,1}$ are the product and fiber salt concentrations at the collocation point along the longitudinal distance of the fiber.

As a further simplification $C_{3,1}$ is taken as $C_p$.

(II) The salt material balance leads to: (18)

$$1=(1-\Delta)C_{1,o}+\Delta C_p$$

(III) The salt diffusion equation at the collocation point is:

$$C_{3,1}\,\overline{V}_w=\theta(C_{1,1}-C_{3,1})$$

which can be further approximated to (19)

$$C_p\,\overline{V}_w=\theta(C_{1,1}-V_p)$$

(IV) The productivity is given by: (20)

$$\Delta=\frac{-(R_o-1)Q}{2}=\frac{1(R_o-1)\phi_1\,\overline{V}_w}{2}$$

For the complete mixing model $C_{1,1}=C_{1,0}$ and these four equations are solved to obtain $V_w$, $C_p$, $C_{1,0}$. For the case of partial mixing $C_{1,1}$ should be substituted by the appropriate relations derived in reference [6].

In reference [6], it has been shown that: (21)

$$C_{1,1}=C_{1,i}+\frac{3}{2}P_e\left(\frac{R_o-1}{R_o+1}\right)N_1$$

where, (22)

$$N_1=\frac{-C_{1,1}\,\overline{R}\,\overline{Q}\left(\frac{R_o+1}{2}\right)}{\left(\frac{-4(R_o+1)}{1+2R_o-3R_o^2}\right)+P_e\,V_{1,1}}$$

and that, (23)

$$C_{1,o}=\frac{4}{3}(C_{1,1}-\frac{1}{4}C_{1,1})$$ (24)

$$V_{1,1}=\frac{2}{1+R_o}\left[1+\frac{Q}{8}\left(R_o^2+2R_o-3\right)\right]$$

Eliminating $C_{1,i}$ between (21, 23), we obtain (25)

$$C_{1,0}-C_{1,1}=\frac{P_e(R_o-1)}{2(R_o+1)}N_1$$

Note that (26)

$$\overline{Q} = \frac{-2\Delta}{R_o^2 - 1}$$

Substituting (22, 26) into (25), we obtain:

$$C_{1,0} - C_p = (C_{1,1} - C_p)\left[1 + \frac{\frac{P_e}{2}\frac{\Delta}{R_o + 1}}{\frac{4(R_o - 1)}{(3R_o + 1)(R_o - 1)} + \frac{2P_e}{(R_o + 1)}\left\{1 - \frac{\Delta}{4}\left(\frac{R_o + 3}{R_o + 1}\right)\right\}}\right] \tag{27}$$

This equation can be used to eliminate ($C_{1,1}$ - Cp) from equations (17, 19), and hence equations (17-20) can be used to determine $C_{1,0}$, $C_p$, $V_w$.

However,

$$C_{1,1} - C_p = \frac{\overline{V}_w C_p}{\theta} = \frac{\overline{Q} C_p}{\phi_1 \theta} = \frac{2\Delta C_p}{\phi_1 (R_o^2 - 1)\upsilon} \tag{28}$$

Thus:

$$C_{1,0} - C_p = \frac{-2\Delta C_p}{\Phi_1 \theta (R_o^2 - 1)}\left[1 + \frac{\frac{P_e}{2}\frac{\Delta}{R_o + 1}}{\frac{4(R_o - 1)}{(3R_o + 1)(R_o - 1)} + \frac{2P_e}{(R_o + 1)}\left\{1 - \frac{\Delta}{4}\left(\frac{R_o + 3}{R_o + 1}\right)\right\}}\right] \tag{29}$$

$$\frac{C_{1,0}}{C_{1,0} - C_p} = 1 - \frac{\Phi_1 \theta (R_o^2 - 1)}{2\Delta} * \tag{30}$$

$$* \frac{\left[\frac{4(R_o + 1)}{(3R_o + 1)(R_o - 1)} + \frac{2P_e}{(R_o + 1)}\left(1 - \frac{\Delta}{4}\frac{(R_o + 3)}{(R_o + 1)}\right)\right]}{\left[\frac{4(R_o + 1)}{(3R_o + 1)(R_o - 1)} + \frac{2P_e}{(R_o + 1)}\left(1 - \frac{\Delta}{4(R_o - 1)}\right)\right]}$$

Noting that the product flowrate Fp satisfies,

$$F_p = \Delta F_F = \Delta v_f \pi \bar{r}_o^2 = \frac{2 \Delta S_m K}{(\Phi_1 \theta (R_o^2 - 1))}$$

then,

$$\frac{C_{1,0}}{C_{1,0} - C_p} = 1 - \frac{S_m K}{F_p} * \tag{31}$$

$$* \frac{\left[ \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{2P_e}{(R_o+1)} \left( 1 - \frac{\Delta}{4} \frac{(R_o+3)}{(R_o+1)} \right) \right]}{\left[ \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{2P_e}{(R_o+1)} \left( 1 - \frac{\Delta}{4(R_o-1)} \right) \right]}$$

where K is the solute diffusion constant. If we substitute $\Delta \cong 1$ in the denominator of equation (31) since ( $\Delta$ /2($R_o$+1) << 1, we obtain

$$\frac{C_{1,0}}{C_{1,0} - C_p} = 1 - \frac{S_m K}{F_p} * \tag{32}$$

$$* \frac{\left[ \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{2P_e}{(R_o+1)} - \frac{P_e (R_o+3) F_p}{2(R_o+1)^2 F_F} \right]}{\left[ \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{2P_e (2R_o+1)}{(R_o+1)^2} \right]}$$

or

$$\frac{C_{1,0}}{C_{1,0} - C_p} = 1 - \frac{1}{\left[ \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{P_e (2R_o+1)}{(R_o+1)^2} \right]} * \tag{33}$$

$$* \left[ \frac{S_m K P_e (R_o+3)}{2F_F (R_o+1)^2} - \left( \frac{4(R_o+1)}{(3R_o+1)(R_o-1)} + \frac{2P_e}{(R_o+1)} \right) \frac{S_m K}{F_p} \right]$$

Thus we are able to show the well established experimental result that the reciprocal of the rejection is linearly related to the reciprocal of the product flow rate. Moreover, this relation shows that the intercept is not 1 (as observed by Soltanieh and Gill) because of the partial mixing (Finite value for $P_e$). Also the intercept depends on the feed flow rate. In addition, the intercept can be used to estimate Peclet number.

However, the Peclet number calculated from equation (33) is very sensitive to the value of intercept. Thus it is better estimated from the rigorous model.

## Parameters Estimation

Soltanieh and Gill presented the data for a module receiving salt water with different concentrations and two flow rates. The results show that K depends on concentration. From their regression analysis, they obtained the following Table 1.

**Table 1. Regression of experimental results of Soltanieh and Gill [2]**

| **Inlet concentration** | **$F_F = 250$ cm$^3$/sec** | | **$F_F = 125$ cm$^3$/sec** | |
|---|---|---|---|---|
| **$C_F$, ppm** | **Intercept a** | **Slope b*10$^6$** | **Intercept a** | **Slope b*10$^6$** |
| 1,000 | 0.995 | 3.145 | 0.982 | 3.421 |
| 5,000 | 0.973 | 8.978 | 0.982 | 8.706 |
| 10,000 | 0.957 | 13.479 | 0.923 | 13.550 |
| 15,000 | 0.966 | 15.021 | 0.925 | 15.236 |
| 20,000 | 0.953 | 15.990 | 0.950 | 14.801 |
| 25,000 | 0.962 | 15.570 | 0.930 | 15.548 |
| 30,000 | 0.957 | 15.727 | 0.871 | 17.198 |
| 35,000 | 0.969 | 16.021 | 0.932 | 16.087 |

These data fit a relation of the form:

$$\frac{C_{1,0}}{C_{1,0}-C_p} = a + b\frac{S_m}{F_p}$$

$$S_m = 1.69578 * 106, \ R_o = 4.2$$

Using the rigorous model (equations (1-16)), we estimated that an average Peclet number of 6 best fits the experimental data of Gill and Soltanieh [2]. Using this value in equation (31) while assuming a value of $\Delta$ = 1 in the denominator, we obtain

$$\frac{C_{1,0}}{C_{1,0}-C_p} = 1 + \frac{S_m K}{F_p}\frac{2.79 - \dfrac{0.8F_p}{F_F}}{2.55} = 1 - \frac{0.314 S_m K}{F_F} + \frac{1.095 S_m K}{F_p} \tag{35}$$

Using the value of b in Table 1, we obtain the following values of K and intercept from equation (35), as presented in Table 2.

**Table 2. K and Intercept estimations**

| | Feed cont. ppm | 1,000 | 5,000 | 10,000 | 15,000 | 20,000 | 25,000 | 30,000 | 35,000 |
|---|---|---|---|---|---|---|---|---|---|
| $F_F = 250$ | $K*10^6$ | 2.84 | 8.10 | 12.15 | 13.53 | 14.45 | 14.04 | 14.22 | 14.48 |
| | Intercept | 0.994 | 0.984 | 0.975 | 0.972 | 0.969 | 0.971 | 0.970 | 0.969 |
| FF = 125 | $K*10^6$ | 3.11 | 7.90 | 12.24 | 13.77 | 13.40 | 14.04 | 15.55 | 14.50 |
| | Intercept | 0.984 | 0.969 | 0.949 | 0.943 | 0.945 | 0.941 | 0.935 | 0.939 |

The intercept calculated compares very well with these estimated from experimental results in Table (1).

## Conclusions

In this paper, we have presented a simple model to calculate the performance of a reverse osmosis hollow fine fiber module. Equations (17-20) and (27) can be used for this purpose. They tend to the complete mixing model [2] as $P_e \rightarrow 0$.

In addition, we obtained a linear relationship between the reciprocal of the rejection and the reciprocal of the products flow rate through an approximation of the model for modules with radial dispersion using one point collocation method. The equation yields a non-unity intercept due to partial mixing, with unity intercept obtained only at complete mixing conditions. Future experimental work should be directed towards correlating the radial dispersion coefficient with module configuration, dimensions and operating conditions.

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# نموذج مبسط لنماذج الألياف الدقيقة المجوفة مع التوزيع القطري

**إبراهيم صالح المعتاز، مصطفى علي سليمان وعبدالله عيضه الزهراني**
*قسم الهندسة الكيميائية، كلية الهندسة، جامعة الملك سعود، ص.ب ٨٠٠،*
*الرياض ١١٤٢١، المملكة العربية السعودية*
(استلم في ١٩٩٦/٣/٢٧م ؛ وقبل للنشر في ١٩٩٧/٣/١م)

**ملخص البحث**. هناك إثباتات معملية كافية تشير إلى أن مقلوب طرد الأملاح لنماذج الألياف الدقيقة المجوفة يرتبط خطيا مع مقلوب معدل سريان المياه المنتجة. لقد أوضحنا أن هذه العلاقة يمكن استنتاجها من النموذج الرياضي المفصل للتوزيع القطري باستخدام تقريب معتمد على تطبيق طريقة التنظيم المتعامل لنقطة واحدة.