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Concerning the Zeros of Solutions of Certain Quasilinear Differential Equations of the Fifth Order

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This paper is concerned with the distribution of zeros of the solutions of certain quasilinear differential equations of the fifth order. The fundamental properties of the solutions of these differential equations are also studied.

We consider the differential equations of the fifth order of the form

(a) $$(p(x)y')'''' + q(x)y = 0$$

and

(b) $$(p(x)z'''')' - q(x)z = 0 ,$$

where $p(x) > 0$ and $q(x) \geqq 0$ are of class$C(-\infty,\infty)$, and $q(x) \equiv 0$ does not hold in any interval.

In this paper we derive the relations between the solutions of the differential equations (a) and (b). We prove that for the equations (a) and (b) there exist solutions without zero points on the infinite half-axis $(-\infty,a)$, (a,∞), $-\infty < a < \infty$ respectively. Then we apply these results to obtain some fundamental properties of the solutions of the equations (a) and (b). Furthermore, assuming that the coefficient $q = q(x,\lambda)$ depends also on a parameter $\lambda \in (\Lambda_1,\Lambda_2)$, a problem concerning the zero points of the solutions will be investigated.

The Relations Between The Solutions

The solutions $y_r(x)$ or $z_r(x)$ $(r=1,2,3,4,5)$ of the differential equations (a) and (b) form a fundamental system of solutions, if their Wronskian

$$W(y_1,y_2,y_3,y_4,y_5)=\begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1' & y_2' & y_3' & y_4' & y_5' \\ (py_1')' & (py_2')' & (py_3')' & (py_4')' & (py_5')' \\ (py_1')'' & (py_2')'' & (py_3')'' & (py_4')'' & (py_5')'' \\ (py_1')''' & (py_2')''' & (py_3')''' & (py_4')''' & (py_5')''' \end{vmatrix}$$

or

$$W^*(z_1,z_2,z_3,z_4,z_5)=\begin{vmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1' & z_2' & z_3' & z_4' & z_5' \\ z_1'' & z_2'' & z_3'' & z_4'' & z_5'' \\ z_1''' & z_2''' & z_3''' & z_4''' & z_5''' \\ z_1'''' & z_2'''' & z_3'''' & z_4'''' & z_5'''' \end{vmatrix}$$

respectively is at least at one point in the interval $(-\infty,\infty)$ different from zero. Obviously

$$W(y_1,y_2,y_3,y_4,y_5)=\frac{c}{p(x)} \quad ,$$

$$W^*(z_1,z_2,z_3,z_4,z_5)=\frac{c}{p(x)} \quad ,$$

where c is a suitable constant depending on the choice of the solutions (Greguš and Abdel Karim 1969).

Let $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ be arbitrary linearly independent solutions of the differential equation (a). Then the function

(1) $$z(x)=w[y_1,y_2,y_3,y_4](x)=$$

$$=\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ (py_1')' & (py_2')' & (py_3')' & (py_4')' \\ (py_1')'' & (py_2')'' & (py_3')'' & (py_4')'' \end{vmatrix}$$

is a solution of the differential equation (b). On the other hand, if $z_1(x), z_2(x), z_3(x), z_4(x)$ are arbitrary linearly independent solutions of the differential equation (b), then the function

(2) $y(x)=w^*[z_1,z_2,z_3,z_4](x)=$

$$=p(x)\begin{vmatrix} z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \\ z_1'' & z_2'' & z_3'' & z_4'' \\ z_1''' & z_2''' & z_3''' & z_4''' \end{vmatrix}$$

is a solution of the differential equation (a).

Moreover, if $z(x)$ is an arbitrary solution of the differential equation (b), then there exist four solutions $y_1(x), y_2(x), y_3(x), y_4(x)$ of the differential equation (a) such that the relation (1) holds; and on the other hand, if $y(x)$ is an arbitrary solution of the differential equation (a), then there exist four solutions $z_1(x), z_2(x), z_3(x), z_4(x)$ of the differential equation (b) such that the relation (2) holds (Abdel Karim, 1973).

If $y_1(x), y_2(x), y_3(x), y_4(x), y_5(x)$ form a fundamental system of solutions of the differential equation (a), then

$$z_1(x)=w[y_2,y_3,y_4,y_5](x), \quad z_2(x)=w[y_1,y_3,y_4,y_5](x),$$

$$z_3(x)=w[y_1,y_2,y_4,y_5](x), \quad z_4(x)=w[y_1,y_2,y_3,y_5](x),$$

$$z_5(x)=w[y_1,y_2,y_3,y_4](x)$$

form a fundamental system of solutions of the differential equation (b). Further, if $z_1(x)$, $z_2(x), z_3(x), z_4(x), z_5(x)$ form a fundamental system of solutions of (b), then

$$y_1(x)=w^*[z_2,z_3,z_4,z_5](x), \quad y_2(x)=w^*[z_1,z_3,z_4,z_5](x),$$

$$y_3(x)=w^*[z_1,z_2,z_4,z_5](x), \quad y_4(x)=w^*[z_1,z_2,z_3,z_5](x),$$

$$y_5(x)=w^*[z_1,z_2,z_3,z_4](x)$$

from a fundamental system of solutions of (a) (Greguš 1965, Greguš and Abdel Karim, 1969).

For the solutions of the differential equation (a) the following integral identities hold

(3) $$y(py')''' - \int_a^x y'(py')''' - qy^2 dt = \text{const.}$$

(4) $$(py')''' + \int_a^x qy dt = \text{const.},$$

$$-\infty < a < \infty, \quad -\infty < x < \infty.$$

The integral identities for the solutions of the differential equation (b) have the form

(3') $$z.pz'''' - \int_a^x (z'.pz'''' + qz^2)dt = \text{const.}$$

(4') $$pz'''' - \int_a^x qz dt = \text{const.}.$$

Solutions Without Zeros

We are going to prove the following theorems:

Theorem 1.

Let y(x) be the solution of the differential equation (a) with the alternative initial values

i) $y(a) = y'(a) = (py')'(a) = (py')''(a) = 0$, $(py')'''(a) \neq 0$

or

ii) $y(a) = y'(a) = (py')'(a) = (py')'''(a) = 0$, $(py')''(a) \neq 0$

or

(iii) $y(a) = y'(a) = (py')''(a) = (py')'''(a) = 0$, $(py')'(a) \neq 0$

or

iv) $y(a) = (py')'(a) = (py')''(a) = (py')'''(a) = 0$, $y'(a) \neq 0$

or

v) $y'(a) = (py')'(a) = (py')''(a) = (py')'''(a) = 0$, $y(a) \neq 0$,

$-\infty < a < \infty$. Then $y(x)$, $y'(x)$, $(py')'(x)$, $(py')''(x)$, $(py')'''(x)$ have no zero point to the left side of a.

Proof.

Let $y(x)$ be the solution of the differential equation (a) satisfying the initial conditions i), and let $(py')'''(a) > 0$. Suppose on the contrary that $(py')'''(x_1) = 0$, where $x_1 < a$ is the first zero point of $(py')'''$ to the left of a. Then the relation

$$\operatorname{sgn} y = -\operatorname{sgn} y' = \operatorname{sgn} (py')' = -\operatorname{sgn} (py')'' = \operatorname{sgn} (py')'''$$

holds in (x_1, a). The integral identity (3) leads to the contradiction

$$0 > [y(py')''']_a^{x_1} - \int_a^{x_1} [y'(py')''' - qy^2]\, dt = 0$$

Then $(py')'''$ has no zero point for $x < a$. From the properties of the monotonic functions it follows that $(py')''$, $(py')'$, y', y have no zero point for $x < a$.

Consider the solution $y(x)$ of the differential equation (a) with the initial values ii), and suppose that $x_1 < a$ is the first zero point of $(py')''$ to the left of a. Then

$$\operatorname{sgn} y = -\operatorname{sgn} y' = \operatorname{sgn} (py')' = -\operatorname{sgn} (py')'' \quad \text{in } (x_1, a).$$

Integration of (4) from a to x_1 gives the contradiction

$$0 \neq [(py')'']_a^{x_1} + \int_a^{x_1} (x_1 - t)\, q(t)\, y(t)\, dt = 0\ ,$$

since y keeps its sign in (x_1, a). Then $(py')''$ and consequently $(py')'$, y', y have no zero point to the left side of a. Using the integral identity (4), we find that $(py')'''$ has no zero point for $x < a$.

Supposing that $y(x)$ is the solution of the differential equation (a) satisfying the initial conditions iii), then

$$\operatorname{sgn} y = -\operatorname{sgn} y' = \operatorname{sgn} (py')' \quad \text{in} (x_1, a),$$

where $x_1 < a$ is assumed to be the first zero point of $(py')'$ to the left side of a. Double integration of (4) gives

$$0 \neq [(py')']_a^{x_1} + \tfrac{1}{2} \int_a^{x_1} (x_1 - t)^2\, q(t)\, y(t)\, dt = 0\ .$$

It follows that $(py')'$ and hence y', y have no zero point for $x < a$. By virtue of (4) and its integration from a to x, we find that $(py')'''$ and $(py')''$ respectively have no zero point for $x < a$.

The other cases can be similarly proved.

Theorem 2.

If $z(x)$ is the solution of the differential equation (b) satisfying the alternative initial conditions

$$\text{i}')\ z(a)=z'(a)=z''(a)=z'''(a)=0,\ z''''(a)\neq 0$$

or

$$\text{ii}')\ z(a)=z'(a)=z''(a)=z''''(a)=0,\ z'''(a)\neq 0$$

or

$$\text{iii}')\ z(a)=z'(a)=z'''(a)=z''''(a)=0,\ z''(a)\neq 0$$

or

$$\text{iv}')\ z(a)=z''(a)=z'''(a)=z''''(a)=0,\ z'(a)\neq 0$$

or

$$\text{v}')\ z'(a)=z''(a)=z'''(a)=z''''(a)=0,\ z(a)\neq 0,$$

$a\in(-\infty,\infty)$, then neither $z(x)$ nor its derivatives $z'(x)$, $z''(x)$, $z'''(x)$, $z''''(x)$ have zero points to the right side of a.

Proof.

Let $z(x)$ be the solution of the differential equation (b) satisfying the initial conditions i'). Then

$$\operatorname{sgn} z=\operatorname{sgn} z'=\operatorname{sgn} z''=\operatorname{sgn} z'''=\operatorname{sgn} z'''' \quad \text{in } (a,x_1),$$

where $x_1>a$ is assumed to be the first zero point of z'''' to the right of a. Using *e.g.* the integral identity (3'), we obtain the contradiction

$$0>[z.\ pz'''']_a^{x_1}-\int_a^{x_1}(z'.pz''''+qz^2)\ dt=0$$

Then z'''' and hence z''', z'', z', z have no zero point for $x>a$.

Using the same procedure, the other cases can be proved.

Fundamental Properties of the Solutions

Using the results of the preceding paragraph, we shall obtain some basic properties of the solutions of the differential equations (a) and (b).

In fact, the following theorems are established:

Theorem 3.

Let $0 < p(x) \leqq \alpha$ for $x \in (-\infty, \infty)$, where α is a constant. Let $y(x)$ be the solution of the differential equation (a) satisfying at the point $a \in (-\infty, \infty)$ the alternative initial conditions i)-v), in which the sign $\neq$ is replaced by $>$. Then in the cases i), iii), v) [ii), iv)] there hold

$$\lim_{x \to -\infty} y(x) = +\infty[-\infty], \qquad \lim_{x \to -\infty} y'(x) = -\infty[+\infty],$$

$$\lim_{x \to -\infty} (py')'(x) = +\infty[-\infty], \ \lim_{x \to -\infty} (py')''(x) = -\infty[+\infty]$$

and there exists also $\lim\limits_{x \to -\infty} (py')'''(x)$ which is finite or $+\infty[-\infty]$.

Proof.

Application of theorem 1 shows that $y, y', (py')', (py')'', (py')'''$ have no zero point for $x < a$. Moreover, there hold in the cases i), iii), v) [ii), iv)]

$$y > 0,\ y' < 0,\ (py')' > 0,\ (py')'' < 0,\ (py')''' > 0 \text{ for } x < a$$

$$[y < 0,\ y' > 0,\ (py')' < 0,\ (py')'' > 0,\ (py')''' < 0 \text{ for } x < a].$$

Let $y(x)$ be the solution of the differential equation (a) satisfying the initial conditions i). The integral identity (4) leads to the following inequalities

$$(py')''(x) < (py')'''(a).(x-a),$$

$$(py')'(x) > \frac{(py')'''(a)}{2\,\alpha}(x-a)^2,$$

$$y'(x) < \frac{(py')'''(a)}{3!\,\alpha}(x-a)^3,$$

$$y(x) > \frac{(py')'''(a)}{4!\,\alpha}(x-a)^4$$

which are valid for $x < a$. It follows from these inequalities that $y \to +\infty, y' \to -\infty$, $(py')' \to +\infty$, $(py')'' \to -\infty$ as $x \to -\infty$. Referring to the differential equation (a), we see that $(py')''' \leqq 0$ for $x < a$, where the equality sign does not hold in any interval.

Therefore $(py')'''$ is a positive non-increasing function in $(-\infty,a)$ and there exists $\lim\limits_{x\to-\infty}(py')'''$.

Consider the solution y(x) of the differential equation (a) with the property ii). Successive integration of (4) from a to x gives for $x<a$

$$(py')''(x)>\int_x^a(x-t)q(t)y(t)dt,$$

$$(py')'(x)<(py')''(a)(x-a),$$

$$y'(x)\frac{(py')''(a)}{2!\,\alpha}(x-a)^2,\quad y(x)\frac{(py')''(a)}{3!\,\alpha}(x-a)^3$$

from which the required limits of y, y', $(py')'$, $(py')''$ follow. Since $(py')''''\geqq 0$ for $x<a$, where the equality sign holds only for the isolated points, then $(py')'''$ is a negative monotonic nondecreasing function for $x<a$ and there exists $\lim\limits_{x\to-\infty}(py')'''$.

Supposing that y(x) is the solution of the differential equation (a) satisfying *e.g.* the initial conditions v), we obtain from (4) the following inequalities for $x<a$

$$(py')''(x)=\int_x^a(x-t)q(t)y(t)dt,\quad (py')'(x)=\tfrac{1}{2}\int_x^a(x-t)^2q(t)y(t)\,dt,$$

$$y'(x)\leqq\frac{1}{3!\,\alpha}\int_x^a(x-t)^3q(t)y(t)\,dt,\quad y(x)>\frac{1}{4!\,\alpha}\int_x^a(x-t)^4q(t)y(t)\,dt,$$

from which the requirement follows.

The other cases can be similarly proved.

Theorem 4.

Let $0<p(x)\leqq\alpha$ for $x\in(-\infty,\infty)$. Let z(x) be the solution of the differential equation (b) satisfying at the point $a\in(-\infty,\infty)$ the alternative initial conditions i') – v'), in which the sign $\neq$ is replaced by $>$. Then there holds

$$\lim_{x\to\infty}z^{(k)}=+\infty\ (\text{for } k=0,1,2,3)$$

and there exists also $\lim\limits_{x\to\infty}z''''$ which is finite or $+\infty$.

Proof.

Referring to theorem 2 it follows that $z^{(k)} > 0$ $(k=0,1,2,3,4)$ for $x > a$. Successive integration of (4′) gives *e.g.* in the cases i′) and iv′) the following inequalities which are valid for $x > a$

$$z'''(x) > \frac{(pz'''')(a)}{\alpha}(x-a), \quad z''(x) > \frac{(pz'''')(a)}{2\alpha}(x-a)^2$$

$$z'(x) > \frac{(pz'''')(a)}{3!\,\alpha}(x-a)^3, \qquad z(x) > \frac{(pz'''')(a)}{4!\,\alpha}(x-a)^4$$

and

$$z'''(x) \geqq \frac{1}{\alpha}\int_a^x (x-t)\, q(t)\, z(t)\, dt, \qquad z''(x) \geqslant \frac{1}{2\alpha}\int_a^x (x-t)^2\, q(t)\, z(t)\, dt.$$

$$z'(x) > \frac{1}{3!\,\alpha}\int_a^x (x-t)^3\, q(t)\, z(t)\, dt, \qquad z(x) > z'(a)\,(x-a)$$

respectively. In all cases, it follows that $z^{(k)}(x) \to +\infty$ $(k=0, 1, 2, 3)$. By using the differential equation (b), the existence of $\lim_{x\to\infty} z''''$ is established.

Theorem 5 (*Uniqueness theorem*).

If $y_r(x)[z_r(x)]$ (for $r = 1,2,3,4$) are linearly independent solutions of the differential equation (a) [(b)], then they can have at most one common zero point.

Proof.

We prove the theorem for the differential equation (a). Suppose on the contrary, that $y_r(x)$ $(r=1,2,3,4)$ have two common zero points at a and b, $b > a$. Let $y(x)$ be the solution of the differential equation (a) satisfying the initial conditions i) at the point b. Evidently $y(x)$ can be written in the form

$$y(x) = \sum_{r=1}^{4} c_r y_r(x),$$

where the constants c_r may be obtained from the equation

$$\sum_1^4 c_r y_r(b) = 0, \qquad \sum_1^4 c_r y_r'(b) = 0,$$

$$\sum_{1}^{4} c_r(py_r')'(b)=0, \qquad \sum_{1}^{4} c_r(py_r')''(b)=0.$$

Certainly at least one of the constants $c_r(r=1,2,3,4)$ does not vanish, since the coefficient determinant

$$\begin{vmatrix} y_1(b) & y_2(b) & y_3(b) & y_4(b) \\ y_1'(b) & y_2'(b) & y_3'(b) & y_4'(b) \\ (py_1')'(b) & (py_2')'(b) & (py_3')'(b) & (py_4')'(b) \\ (py_1')''(b) & (py_2')''(b) & (py_3')''(b) & (py_4')''(b) \end{vmatrix}$$

is equal to zero. Therefore the solution $y(x)$ of the differential equation (a) satisfies the initial conditions i) at the point b and has the simple zero point at the point $a<b$, which is a contradiction with theorem 1.

The statement for the differential equation (b) can be similarly proved. Further, we state (Abdel Karim, 1973).

Theorem 6 (*Separation theorem*).

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the differential equation (a) with a common zero of order 3 at the point $a\in(-\infty,\infty)$, then the zeros of $y_1(x)$ and $y_2(x)$ separate each other in (a,∞).

On The Zeros of the Solutions

In this paragraph we consider the differential equation

(5) $\qquad (py')''''+q(x,\lambda)y=0,$

where $p(x)>0$ is a continuous function of $x\in(-\infty,\infty)$ and $q(x,\lambda)\geqq 0$ is a continuous function of $x\in(-\infty,\infty)$ and $\lambda\in(\Lambda_1,\Lambda_2)$, and $q\not\equiv 0$ in any interval.

Then the following theorem holds.

Theorem 7.

Let $\lim_{\lambda\to\Lambda_2} q(x,\lambda)=+\infty$ hold uniformly for all $x\in(-\infty,\infty)$. Let $-\infty<a<b<\infty$. If $y(x,\lambda)$ is the solution of the differential equation (5) with the alternative initial values i) $-$ v) at the point a, then there exists a parameter $\lambda\in(\Lambda_1,\Lambda_2)$ such that $y(x,\lambda)$ has a farther zero point in (a,b).

Proof.

Let $y(x)$ be the solution of the differential equation (5) with the initial conditions i). We compare the differential equation (5) with the equation $(py')''''=0$, whose fundamental system of solutions is

$$Y_1 = \frac{1}{3!}\int_a^x \frac{(t-a)^3}{p(t)}\,dt, \qquad Y_2 = \frac{1}{2}\int_a^x \frac{(t-a)^2}{p(t)}\,dt,$$

$$Y_3 = \int_a^x \frac{t-a}{p(t)}\,dt, \qquad Y_4 = \int_a^x \frac{1}{p(t)}\,dt, \qquad Y_5 = 1$$

and their Wronski determinant is equal to $W(x) = \frac{1}{p(x)}$. Without loss of generality let $(py')''' = 1$. Since Y_1 satisfies at the point a the same initial conditions as y(x), then by means of the method of variation of constants, y can be written in the form (see Greguš, 1963)

$$(6) \qquad y(x,\lambda) = Y_1(x) - \int_a^x q(t,\lambda)\frac{W(x,t)}{W(t)}y(t,\lambda)\,dt =$$

$$= \frac{1}{3!}\int_a^x \frac{(t-a)^3}{p(t)}\,dt - \int_a^x q(t,\lambda)\ p(t)\ W(x,t)\ y(t,\lambda)\,dt.$$

The function W(x,t) is of the form

$$(7)\, W(x,t) = \begin{vmatrix} Y_1(x) & Y_2(x) & Y_3(x) & Y_4(x) & Y_5(x) \\ Y_1(t) & Y_2(t) & Y_3(t) & Y_4(t) & Y_5(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) & Y_4'(t) & Y_5'(t) \\ (pY_1')'(t) & (pY_2')'(t) & (pY_3')'(t) & (pY_4')'(t) & (pY_5')'(t) \\ (pY_1')''(t) & (pY_2')''(t) & (pY_3')''(t) & (pY_4')''(t) & (pY_5')''(t) \end{vmatrix}$$

Evidently for fixed t the function $W(x,t) = Y^*(x)$ is a solution of the differential equation $(pY')'''' = 0$ with the properties $Y^*(t) = Y^{*\prime}(t) = (pY^{*\prime})'(t) = (pY^{*\prime})''(t) = 0$, $(pY^{*\prime})'''(t) = W(t) = \frac{1}{p(t)} > 0$, and therefore

$$W(x,t) = \frac{1}{3!p(t)}\int_t^x \frac{(s-t)^3}{p(s)}\,ds.$$

Substituting in (6), we obtain

$$(8) \qquad y(x,\lambda) = \frac{1}{3!}\int_a^x \frac{(t-a)^3}{p(t)}\,dt - \frac{1}{3!}\int_a^x q(t,\lambda)\,y(t,\lambda)\left(\int_t^x \frac{(s-t)^3}{p(s)}\,ds\right)dt.$$

Suppose on the contrary, that $y(x,\lambda)$ has no zero point for $x\in(a,b)$ and $\lambda\in(\Lambda_1,\Lambda_2)$. Then $y(x,\lambda)>0$ for $a<x<b$ and $\Lambda_1<\lambda<\Lambda_2$. But $y(b,\lambda)$ is a continuous function in $\lambda\in(\Lambda_1,\Lambda_2)$ and with increasing $\lambda\to\Lambda_2$ $y(b,\lambda)$ (see(8)) will be negative. Hence the proof is complete.

Suppose that $y(x)$ is the solution of the differential equation (5) satisfying *e.g.* the initial conditions iv), and let $y'(a)=1$. We compare with the equation $(pY')''''=0$, which has a fundamental system of solutions

$$Y_1=\frac{1}{3!}\int_a^x\frac{(t-a)^3}{p(t)}\,dt,\qquad Y_2=\frac{1}{2}\int_a^x\frac{(t-a)^2}{p(t)}\,dt\ ,$$

$$Y_3=\int_a^x\frac{t-a}{p(t)}\,dt,\qquad Y_4=p(a)\int_a^x\frac{1}{p(t)}\,dt,\qquad Y_5=1,$$

whose Wronskian is equal to $W(x)=\dfrac{p(a)}{p(x)}$. It is obvious that Y_4 and y satisfy the same initial conditions at the point a. Analogous to (6), y can be written as

$$y(x,\lambda)=Y_4(x)-\int_a^x q(t,\lambda)\,\frac{W(x,t)}{W(t)}\,y(t,\lambda)\,dt=$$

$$=p(a)\int_a^x\frac{1}{p(t)}\,dt-\frac{1}{p(a)}\int_a^x q(t,\lambda)\,p(t)\,W(x,t)\,y(t,\lambda)\,dt,$$

where $W(x,t)$ is defined in (7). Evaluating $W(x,t)$ and substituting in the preceding equation, we get

$$y(x,\lambda)=p(a)\int_a^x\frac{1}{p(t)}\,dt-\frac{1}{3!}\int_a^x q(t,\lambda)\,y(t,\lambda)\left(\int_t^x\frac{(s-t)^3}{p(s)}\,ds\right)dt\ ,$$

from which the requirement follows.
By the same way, the other cases can be proved.

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أصفار حلول معادلات تفاضلية معينة شبه خطية من الرتبة الخامسة

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