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Some Properties of Quasilinear Differential Equations of the Third Order

Rahmi Ibrahim Ibrahim Abdel Karim
Department of Mathematics, Faculty of Science, Riyad University, Riyad, Saudi Arabia.

In this paper certain quasilinear differential equations of the third order will be studied. Then some fundamental properties of the solutions of these differential equations will be derived.

1. In this paper we consider the differential equations of the third order of the form

$$(1) \qquad (p(x)y')'' + q(x)y = 0$$

and

$$(2) \qquad (p(x)z'')' - q(x)z = 0,$$

where $p(x) > 0$, $q(x) \geqq 0$ are continuous functions of $x \in (-\infty, \infty)$ and $q(x) = 0$ holds only for the isolated points.

Let y_1, y_2 be two arbitrary linearly independent solutions of the differential equation (1). Then the function

$$z(x) = p(x) \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is a solution of the differential equation (2).

Furthermore, if z_1, z_2 are two linearly independent solutions of the differential equation (2), then the function

$$y(x) = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix}$$

is a solution of the differential equation (1). (Greguš, 1965).

If y_1, y_2, y_3 form a fundamental system of solutions of the differential equation (1), then

$$z_1 = p(x) \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}, \quad z_2 = p(x) \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix},$$

$$z_3 = p(x) \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

form a fundamental system of solutions of the differential equation (2). On the other hand, if z_1, z_2, z_3 form a fundamental system of solutions of the differential equation (2), then

$$y_1 = \begin{vmatrix} z_2 & z_3 \\ z_2' & z_3' \end{vmatrix}, \quad y_2 = \begin{vmatrix} z_1 & z_3 \\ z_1' & z_3' \end{vmatrix},$$

$$y_3 = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix}$$

form a fundamental system of solutions of the differential equation (1). (Birkhoff, 1910-1911; Greguš and Abdel Karim, 1973).

Theorem 1.

The solutions of the differential equation (1) with the property

$$y(a) = y'(a) = 0, \ (py')'(a) \neq 0, \ a \in (-\infty, \infty)$$

are linearly dependent.

Proof.

Let y(x) be an arbitrary solution of the differential equation (1) with the property

$$y(a)=y'(a)=0 \quad , \qquad (py')'(a)=k\neq 0.$$

Then y(x) can be written in the form

$$y(x)=\sum_{1}^{3} c_r y_r(x), \tag{3}$$

where $y_1(x)$, $y_2(x)$, $y_3(x)$ form a fundamental system of solutions of the differential equation (1), and the constants c_r are obtained from the equations

$$\sum_{1}^{3} c_r y_r(a)=0, \quad \sum_{1}^{3} c_r y_r'(a)=0, \quad \sum_{1}^{3} c_r (py_r')'(a)=k\neq 0$$

Evaluating the constants c_1, c_2, c_3, and substituting in (3), we get

$$y(x)=\frac{k}{(p\bar{y}')'(a)}\bar{y}(x) \ ,$$

where

$$\bar{y}(x)= \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1(a) & y_2(a) & y_3(a) \\ y_1'(a) & y_2'(a) & y_3'(a) \end{vmatrix}$$

is a solution of the differential equation (1) with the property

$$\bar{y}(a)=\bar{y}'(a)=0, \quad (p\bar{y}')'(a)\neq 0.$$

This completes the proof.

Similarly, it can be proved, that the solutions of the differential equation (1) with the property

$$y(a)=(py')'(a)=0 \ , \qquad y'(a)\neq 0$$

or

$$y'(a)=(py')'(a)=0 \ , \qquad y(a)\neq 0; \ a\in(-\infty,\infty)$$

are linearly dependent. (Greguš, 1963).

Analogously, there holds

Theorem 2.

The solutions of the differential equation (2) with the property

$$z(a)=z'(a)=0\ ,\qquad z''(a)\neq 0$$

or

$$z(a)=z''(a)=0\ ,\qquad z'(a)\neq 0$$

or

$$z'(a)=z''(a)=0\ ,\quad z(a)\neq 0;\quad a\in(-\infty,\infty)$$

are linearly dependent.

2. In this paragraph we are going to prove the following theorems:

Theorem 3.

Let y(x) be the solution of the differential equation (1) satisfying the alternative initial conditions

$$\text{i)}\qquad y(a)=y'(a)=0,\qquad (py')'(a)\neq 0$$

or

$$\text{ii)}\qquad y(a)=(py')'(a)=0,\qquad y'(a)\neq 0$$

or

$$\text{iii)}\qquad y'(a)=(py')'(a)=0,\qquad y(a)\neq 0,$$

$-\infty<a<\infty$. Then y(x), y'(x), (py')'(x) have no zero point to the left side of a.

Proof.

Let y(x) be the solution of the differential equation (1) satisfying the initial conditions

$$y(a)=y'(a)=0,\quad (py')'(a)>0,\quad -\infty<a<\infty.$$

Suppose on the contrary that $(py')'(x_1)=0$, where $x_1<a$ is the first zero point of $(py')'$ to the left of a. Integrating the differential equation (1) from a to x_1, we get

$$[(py')']_a^{x_1} - \int_{x_1}^{a} qy \; dt = 0, \tag{4}$$

which is a contradiction, since the left hand side of (4) is negative. From the properties of the monotonic functions it follows also that y', y have no zero point for $x<a$.

The other cases can be similarly proved.

Theorem 4.

If $z(x)$ is the solution of the differential equation (2) with the alternative properties

i') $z(a)=z'(a)=0$, $z''(a)\neq 0$

or

ii') $z(a)=z''(a)=0$, $z'(a)\neq 0$

or

iii') $z'(a)=z''(a)=0$, $z(a)\neq 0$,

$a\in(-\infty,\infty)$, then neither $z(x)$ nor its derivatives $z'(x)$, $z''(x)$ have zero points to the right side of a.

Proof.

Let $z(x)$ be the solution of the differential equation (2) with the property

$$z(a)=z''(a)=0, \qquad z'(a)>0, \qquad -\infty<a<\infty.$$

Suppose that $z'(x_1)=0$, where $x_1>a$ is the first zero point of z' to the right of a. Then

$$\operatorname{sgn} z = \operatorname{sgn} z' \qquad \text{in } (a, x_1).$$

Multiplying (2) with z' and integrating it from a to x_1 we get the contradiction

$$[z'(pz'')]_a^{x_1} - \int_a^{x_1} (qzz' + pz''^2)dt = 0, \tag{5}$$

since the left hand side of (5) is negative, because the expression under the integral sign does not change its sign in (a, x_1). Hence $z'$and consequently z have no zero point to the right side of a.

By using the integration of (2), it can be also shown that z'' has no zero point for $x > a$.

Analogously the other cases can be proved.

We note, that other similar types of theorems are given by the author (1977).

3. By using the results of paragraph 2, we shall prove some properties of the solutions of the differential equations (1) and (2).

Theorem 5.

Let $0 < p(x) \leqq m$ for $x \in (-\infty, \infty)$, where m is a constant. Let y(x) be the solution of the differential equation (1) satisfying at the point $a \in (-\infty, \infty)$ the alternative initial conditions i) or ii) or iii), in which the sign $\neq$ is replaced by $>$. Then in the cases i), iii) [(ii)] there hold

$$\lim_{x \to -\infty} y(x) = +\infty \qquad [-\infty],$$

$$\lim_{x \to -\infty} y'(x) = -\infty \qquad [+\infty],$$

and there exists also $\lim_{x \to -\infty} (py')'$ which is finite or $+\infty [-\infty]$.

Proof.

By virtue of theorem 3, it follows that y, y', $(py')'$ have no zero point for $x < a$. Furthermore, there holds in the cases i), iii) [ii)]

$$y > 0 \ , \ y' < 0 \ , \ (py')' > 0 \quad \text{for } x < a$$

$$[y < 0 \ , \ y' > 0 \ , \ (py')' < 0 \quad \text{for } x < a].$$

Case i).

Let y(x) be the solution of the differential equation (1) satisfying the initial conditions

$$y(a) = y'(a) = 0, \ (py')'(a) > 0, \ a \in (-\infty, \infty).$$

Successive integration of the differential equation (1) leads to the following inequalities

$$y'(x) < \frac{(py')'(a)}{m}(x-a) \ , \quad y(x) > \frac{(py')'(a)}{2m}(x-a)^2,$$

which are valid for $x<a$.. It follows from these inequalities that $y \to +\infty, y' \to -\infty$ as $x \to -\infty$.

From the differential equation (1) we see that $(py')' \leqq 0$ for $x<a$, where the equality sign does not hold in any interval. Therefore $(py')'$ is a positive non-increasing function in $(-\infty,a)$ and there exists $\lim\limits_{x \to -\infty} (py')'$.

Case ii).

Consider the solution $y(x)$ of the differential equation (1) with the initial values

$$y(a)=(py')'(a)=0, \quad y'(a)>0, \quad -\infty<a<\infty.$$

Integration of the differential equation (1) from a to x gives for $x<a$

$$y'(x)>\frac{1}{m}\int_x^a (x-t)q(t)y(t)dt, \quad y(x)<\frac{(py')(a)}{m}(x-a).$$

From these two inequalities it is evident that $y(x) \to -\infty$, $y'(x) \to +\infty$ as $x \to -\infty$.

Since $(py')'' \geqq 0$ for $x<a$, where the equality sign holds only for the isolated points, then $(py')'$ is a negative monotonic non-decreasing function for $x<a$ and there exists $\lim\limits_{x \to -\infty} (py')'$.

The remaining case can be similarly proved.

Theorem 6.

Let $0<p(x) \leqq m$ for $x \in (-\infty,\infty)$, where m is a constant. Let $z(x)$ be the solution of the differential equation (2) satisfying at the point $a \in (-\infty,\infty)$ the alternative initial conditions i') or ii') or iii'), in which the sign $\neq$ is replaced by $>$. Then there holds

$$\lim_{x \to \infty} z(x) = \lim_{x \to \infty} z'(x) = +\infty,$$

and there exists also $\lim\limits_{x \to \infty} z''$ which is finite or $+\infty$

Proof.

Application of theorem 4 shows that

$$z>0, \quad z'>0, \quad z''>0 \quad \text{for } x>a.$$

Let z(x) be the solution of the differential equation (2) satisfying the initial conditions i′) or ii′) or iii′). Then the differential equation (2) leads respectively to the following inequalities which are valid for $x > a$

$$z'(x) > \frac{(pz'')(a)}{m}(x-a)\,, \quad z(x) > \frac{(pz'')(a)}{2m}(x-a)^2$$

or

$$z'(x) > \frac{1}{m}\int_a^x (x-t)q(t)z(t)dt, \quad z(x) > z'(a)(x-a)$$

or

$$z'(x) > \frac{1}{m}\int_a^x (x-t)q(t)z(t), \qquad z(x) > \frac{1}{2m}\int_a^x (x-t)^2 q(t)z(t)dt$$

respectively. It follows in all cases that $z \to +\infty$, $z' \to +\infty$ as $x \to +\infty$. Using the differential equation (2), the existence of $\lim_{x\to\infty} z''$ is established.

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بعض الخواص لمعادلات تفاضلية شبه خطية من الرتبة الثالثة

رحمي إبراهيم إبراهيم عبد الكريم

قسم الرياضيات ، كلية العلوم ، جامعة الرياض ، الرياض ، المملكة العربية السعودية

في هذا البحث درست معادلات تفاضلية معينة شبه خطية من الرتبة الثالثة ، ثم اشتقت بعض الخواص الرئيسية لحلول هذه المعادلات التفاضلية .