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Some Boundary Value Problems of The Third Order in Three Points

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In this paper some types of boundary value problems in three points for the solutions of the differential equation of the third order

(a) $$(p(x)y')'' + q(x,\lambda)y = 0$$

will be solved. For the coefficients of (a) it is assumed that $p(x) > 0$ and $q(x,\lambda) \geqq 0$ are continuous functions of $x \in (-\infty,\infty)$ and $\lambda \in (\Delta_1,\Delta_2)$; and $q \equiv 0$ does not hold in any interval. The boundary value problems will be solved by means of the properties of the so-called bands of solutions of the first, second and third kind of the differential equation (a), and the oscillation theorem for the corresponding band of solutions.

We consider the differential equation of the third order of the form

(a) $$(p(x)y')'' + q(x,\lambda)y = 0,$$

where $p(x) > 0$ is a continuous function of $x \in (-\infty,\infty)$ and $q(x,\lambda) \geqq 0$ is a continuous function of $x \in (-\infty,\infty)$ and $\lambda \in (\Lambda_1,\Lambda_2)$; and $q(x,\lambda)$ does not vanish identically in any arbitrary interval.

In a previous paper the author (1976) proved the existence, under certain conditions, of non-trivial solutions of the differential equation (a) satisfying the boundary conditions in three points

$$(1)\quad \begin{cases} y(a,\lambda)=0 \\ \alpha(\lambda)y(b,\lambda)-\beta(\lambda)y'(b,\lambda)=0 \\ \gamma(\lambda)y(c,\lambda)-\delta(\lambda)y'(c,\lambda)=0 \end{cases}$$

where $a \leqq b < c \in (-\infty,\infty)$ are given numbers.

In this paper 8 types of boundary value problems for the solutions of (a) with boundary conditions similar to (1) will be solved. Further 18 types of boundary value problems with the preceding boundary conditions, but in which its first condition $y(a,\lambda)=0$ is replaced either by $y'(a,\lambda)=0$ $('=\frac{d}{dx})$, or by $(py')'(a,\lambda)=0$ will be also studied. For this purpose the properties of the solutions of the differential equation (a) and also of the bands of solutions of the first, second and third kind of (a) will be derived, and the oscillation theorem for these bands of solutions will be establised.

Let y_1 and y_2 be two arbitrary linearly independent solutions of the differential equation (a). Then

$$z=p(x)\,(y_1y_2'-y_1'y_2)$$

is a solution of the differential equation

$$\text{(b)}\qquad (p(x)z'')'-q(x,\lambda)z=0$$

On the other hand, if z_1 and z_2 are two arbitrary linearly independent solutions of the differential equation (b), then the function

$$y=z_1z_2'-z_1'z_2$$

is a solution of the differential equation (a). (Ggreguš, 1965).

It will be assumed that the coefficients p and q in the differential equation (b) have the same properties as stated for the differential equation (a).

If y_1, y_2, y_3 form a fundamental set of solutions of the differential equation (a), then

$$\text{(2)}\qquad z_1=p(y_1y_2'-y_1'y_2),\qquad z_2=p(y_1y_3'-y_1'y_3),$$

$$z_3=p(y_2y_3'-y_2'y_3)$$

form a fundamental set of solutions of (b). Moreover, if z_1, z_2, z_3 form a fundamental set of solutions of (b), then

(3) $$y_1 = z_1 z_2' - z_1' z_2 \qquad y_2 = z_1 z_3' - z_1' z_3 ,$$

$$y_3 = z_2 z_3' - z_2' z_3$$

form a fundamental set of solutions of (a). (Sansone, 1948; Greguš, 1963).

Theorem 1

If z is the solution of the differential equation (b) with the alternative initial values

$$z(a,\lambda) = z'(a,\lambda) = 0, \quad z''(a,\lambda) \neq 0,$$

or

$$z(a,\lambda) = z''(a,\lambda) = 0, \qquad z'(a,\lambda) \neq 0,$$

or

$$z'(a,\lambda) = z''(a,\lambda) = 0, \qquad z(a,\lambda) \neq 0,$$

$-\infty < a < \infty$, then neither z nor its derivatives z', z'' have zero points to the right side of a.

For the proof see (Abdel Karim, 1976).

The Bands of Solutions

Definition

Let y_1, y_2, y_3 be a fundamental set of solutions of the differential equation (a) with the properties

(4) $$\begin{cases} y_1(a,\lambda) = y_1'(a,\lambda) = 0, \ (py_1')'(a,\lambda) \neq 0, \\ y_2(a,\lambda) = (py_2')'(a,\lambda) = 0, \ y_2'(a,\lambda) \neq 0, \\ y_3'(a,\lambda) = (py_3')'(a,\lambda) = 0, \ y_3(a,\lambda) \neq 0; -\infty < a < \infty. \end{cases}$$

The set of solutions of the differential equation (a)

$$y = c_1 y_1 + c_2 y_2, \text{ or } y = c_1 y_1 + c_2 y_3, \text{ or } y = c_1 y_2 + c_2 y_3$$

(c_1, c_2 are arbitrary constants) with the property that $y(a,\lambda)=0$ or $y'(a,\lambda)=0$, or $(py')'(a,\lambda)=0$ is said to be the band of solutions of the first, or second, or third kind of the differential equation (a) at the point a.

Lemma 1

Every band of solutions of the differential equation (a) fulfils the differential equation of the second order

$$\text{(c)} \qquad w(py')' - pw'y' + pw''y = 0,$$

where w is the solution of the differential equation (b) and is equal to $w_1 = z_1$, or $w_2 = z_2$, or $w_3 = z_3$ (see (2)) in the case of the band of the first, or second, or third kind respectively. (Greguš and Abdel Karim, 1969; Abdel Karim, 1973).

Referring to (2) and (4), we see that w_1, w_2, w_3 have the following properties

$$w_1(a,\lambda) = w_1'(a,\lambda) = 0, \quad w_1''(a,\lambda) \neq 0,$$

$$\text{(5)} \qquad w_2(a,\lambda) = w_2''(a,\lambda) = 0, \quad w_2'(a,\lambda) \neq 0,$$

$$w_3'(a,\lambda) = w_3''(a,\lambda) = 0, \quad w_3(a,\lambda) \neq 0; -\infty < a < \infty.$$

Hence, applying theorem 1, it follows that w_r (for r = 1, 2, 3) and its derivatives w_r', w_r'' have no zero point for $x > a$.

Remark

The zero points of every band of solutions are simple in (a, ∞) and separate (if there exist) each other, since every band satisfies a differential equation of the form (c).

We state the following two theorems (Abdel Karim, 1976):

Theorem 2 (*Separation theorem*)

If y_1 and y_2 are two linearly independent solutions of the differential equation (a) with the alternative properties

$$y_1(a,\lambda) = y_2(a,\lambda) = 0,$$

or

$$y_1'(a,\lambda) = y_2'(a, \lambda) = 0,$$

or

$$(py_1')'(a,\lambda) = (py_2')'(a,\lambda) = 0,$$

$-\infty < a < \infty$, then the zero points of y_1 and y_2 separate each other in (a,∞).

Theorem 3

If one solution of the differential equation (a) is oscillatory in some interval (χ,∞), $-\infty < \quad < \infty$, and y is any solution of the differential equation (a) with the alternative properties that $y(a,\lambda)=0$, or $y'(a,\lambda)=0$, or $(py')'(a,\lambda)=0, -\infty < a < \infty$, then y is also oscillatory in (χ,∞).

Referring to the definition of the band of solutions and using theorem 3 (see also theorem 2), it follows

Theorem 4

If the solutions of the band of the first kind of the differential equation (a) at the point a are oscillatory for $x > a$, then the solutions of the band of the second and third kind of the differential equation (a) are also oscillatory for $x > a$.

By means of theorem 4 and the oscillation theorem for the band of solutions of the first kind (Abdel Karim, 1973), we can generalize the oscillation theorem for the band of solutions of the second and third kind.

Hence we obtain

Theorem 5 (*Oscillation theorem*)

Let $\lim\limits_{\lambda \to \Lambda_2} q(x,\lambda) = +\infty$ hold uniformly for all $x \in (-\infty,\infty)$. Let $a < b \in (-\infty,\infty)$ be given numbers, and let $y(x,\lambda)$ be the solution of the differential equation (a) with the alternative properties that $y(a,\lambda)=0$, or $y'(a,\lambda)=0$, or $(py')'(a,\lambda)=0$.

Then for every arbitrary natural number $N>0$ there exists a parameter $\lambda_N \in (\Lambda_1,\Lambda_2)$, so that for $\lambda > \lambda_N$ the solution $y(x,\lambda)$ has in (a,b) at least N zero points, where with increasing $\lambda \to \Lambda_2$ the number of zeros of the solution $y(x,\lambda)$ in the interval (a,b) tends to infinity, and at the same time the distance between two neighbouring zero points tends to zero.

Boundary Value Problem in Three Points

First we state the following

Lemma 2

Let $y(x,\lambda)$ be a non-trivial solution of the differential equation (a) with the alternative properties that $y(a,\lambda)=0$, or $y'(a,\lambda)=0$, or $(py')'(a,\lambda)=0, -\infty < a < \infty$. Then the zero points of $y(x,\lambda)$, $y'(x,\lambda)$, $(py')'(x,\lambda)$ to the right of a (if there exist) are continuous functions of the parameter $\lambda\in(\Lambda_1,\Lambda_2)$ for $x>a$. (Greguš, 1963).

We are going to prove now the following main theorem:

Theorem 6 (*Boundary value problems in three points*)

Let $\lim_{\lambda\to\Lambda_2} q(x,\lambda)=+\infty$ hold uniformly for all $x\in(-\infty,\infty)$. Let a, b, c be given numbers, where $-\infty<a<b<c<\infty$. Further let $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$, $\delta(\lambda)$ be continuous functions of the parameter $\lambda\in(\Lambda_1,\Lambda_2)$, for which $|\alpha|+|\beta|=0$, $|\gamma|+|\delta|=0$, where either $\delta(\lambda)\equiv 0$ or $\delta(\lambda)\neq 0$ for any $\lambda\in(\Lambda_1,\Lambda_2)$.

Then there exists a natural number N and a sequence of parameters (eigenvalues):

$$\lambda_N, \lambda_{N+1}, \ldots, \lambda_{N+p}, \ldots,$$

to which belongs the sequence of functions (eigenfunctions):

$$y_N, y_{N+1}, \ldots, y_{N+p}, \ldots,$$

with the property that $y_{N+p} = y(x,\lambda_{N+p})$ is the solution of the differential equation (a), which fulfils the following alternative boundary conditions:

$$(6)\quad \begin{cases} y(a,\lambda)=0 \\ \alpha(\lambda)\ y(b,\lambda)-\beta(\lambda)\ y'(b,\lambda)=0 \\ \gamma(\lambda)\ y(c,\lambda)-\delta(\lambda)\ (py')'(c,\lambda)=0, \end{cases}$$

or

$$(7)\quad \begin{cases} y(a,\lambda)=0 \\ \alpha(\lambda)\ y(b,\lambda)-\beta(\lambda)\ y'(b,\lambda)=0 \\ \gamma(\lambda)\ y'(c,\lambda)-\delta(\lambda)\ (py')'(c,\lambda)=0 \end{cases}$$

or

$$(8)\quad \begin{cases} y(a,\lambda)=0 \\ \alpha(\lambda)\ y(b,\lambda)-\beta(\lambda)\ (py')'(b,\lambda)=0 \\ \gamma(\lambda)\ y(c,\lambda)-\delta(\lambda)\ y'(c,\lambda)=0, \end{cases}$$

or

$$(9)\qquad \left\{\begin{aligned} & y(a,\lambda)=0 \\ & \alpha(\lambda)\; y(b,\lambda)-\beta(\lambda)\; (py')'(b,\lambda)=0 \\ & \gamma(\lambda)\; y(c,\lambda)-\delta(\lambda)\; (py')'(c,\lambda)=0 \end{aligned}\right.$$

or

$$(10)\qquad \left\{\begin{aligned} & y(a,\lambda)=0 \\ & \alpha(\lambda)\; y(b,\lambda)-\beta(\lambda)\; (py')'(b,\lambda)=0 \\ & \gamma(\lambda)\; y'(c,\lambda)-\delta(\lambda)\; (py')'(c,\lambda)=0, \end{aligned}\right.$$

or

$$(11)\qquad \left\{\begin{aligned} & y(a,\lambda)=0 \\ & \alpha(\lambda)\; y'(b,\lambda)-\beta(\lambda)\; (py')'(b,\lambda)=0 \\ & \gamma(\lambda)\; y(c,\lambda)-\delta(\lambda)\; y'(c,\lambda)=0, \end{aligned}\right.$$

or

$$(12)\qquad \left\{\begin{aligned} & y(a,\lambda)=0 \\ & \alpha(\lambda)\; y'(b,\lambda)-\beta(\lambda)\; (py')'(b,\lambda)=0 \\ & \gamma(\lambda)\; y(c,\lambda)-\delta(\lambda)\; (py')'(c,\lambda)=0, \end{aligned}\right.$$

or

$$(13)\qquad \left\{\begin{aligned} & y(a,\lambda)=0 \\ & \alpha(\lambda)\; y'(b,\lambda)-\beta(\lambda)\; (py')'(b,\lambda)=0 \\ & \gamma(\lambda)\; y'(c,\lambda)-\delta(\lambda)\; (py')'(c,\lambda)=0. \end{aligned}\right.$$

Proof

We prove the boundary value problem *e.g.* with the boundary condition (10). Let $y=c_1y_1+c_2y_2$ be the band of solutions of the first kind of the differential equation (a) at the point a, where y_1 and y_2 satisfy the first and second relations of (4) respectively. Evidently the first boundary condition of (10) is satisfied for every solution of this band. We choose c_1 and c_2 such that the second boundary condition of (10) is satisfied for every $\lambda\in(\Lambda_1,\Lambda_2)$. This is possible, since it is sufficient to choose c_1 and c_2 so that the relations

$$c_1y_1(b,\lambda)+c_2y_2(b,\lambda)=\beta(\lambda)$$

$$c_1(py_1')'(b,\lambda)+c_2(py_2')'(b,\lambda)=\alpha(\lambda)$$

hold. The coefficient determinant of this system is equal to

$$\begin{vmatrix} y_1(b,\lambda) & y_2(b,\lambda) \\ (py_1')'(b,\lambda) & (py_2')'(b,\lambda) \end{vmatrix} = w_1'(b,\lambda) \neq 0$$

(see theorem 1), and at least one of the numbers $\alpha(\lambda), \beta(\lambda)$ is different from zero for $\lambda \in (\Lambda_1,\Lambda_2)$.

For $\lambda = \bar{\lambda} \in (\Lambda_1,\Lambda_2)$ let $y'(x,\bar{\lambda})$ have in (a,c) exactly N zero points. For the N-th and (N+1)-th zero points of $y'(x,\bar{\lambda})$ there holds the inequality

$$x_N(\bar{\lambda}) < c \leqq x_{N+1}(\bar{\lambda}).$$

From the oscillation theorem for the band of solutions of the first kind, it follows that there exists such a parameter $\bar{\bar{\lambda}} > \bar{\lambda} \in (\Lambda_1,\Lambda_2)$, for which $x_{N+1}(\bar{\bar{\lambda}})$ c, where $x_{N+1}(\lambda) = c$ does not hold for any $\lambda > \bar{\bar{\lambda}}$.

From the continuity of $x_{N+1}(\lambda)$ in the parameter $\lambda \in (\Lambda_1,\Lambda_2)$ (see lemma 2), it follows the existence of a largest parameter $\bar{\lambda}_N^* \in [\bar{\lambda},\bar{\bar{\lambda}})$, for which $y'(c,\bar{\lambda}_N^*) = 0$; and therefore $y'(x,\bar{\lambda}_N^*)$ has in (a, c) exactly N zero points. For $\lambda = \bar{\lambda}_N^*$ there holds

$$x_{N+1}(\bar{\lambda}_N^*) = c < x_{N+2}(\bar{\lambda}_N^*).$$

Further, it follows again from the oscillation theorem the existence of such a parameter $\lambda^* > \bar{\lambda}_N^* \in (\Lambda_1,\Lambda_2)$, so that $x_{N+2}(\lambda^*) < c$, and $x_{N+2}(\lambda) = c$ does not hold for any $\lambda > \lambda^*$.

Since $x_{N+2}(\lambda)$ is a continuous function of the parameter $\lambda \in (\Lambda_1,\Lambda_2)$, then there exists in the interval $(\bar{\lambda}_N^*, \lambda^*)$ a smallest parameter $\bar{\lambda}_{N+1}$ and a largest parameter $\bar{\lambda}_{N+1}^*$ for which $y'(c,\bar{\lambda}_{N+1}) = y'(c,\bar{\lambda}_{N+1}^*) = 0$; and hence $y'(x,\bar{\lambda}_{N+1})$ and $y'(x,\bar{\lambda}_{N+1}^*)$ have in (a,c) exactly N+1 zero points.

Continuing in the same way, we find a sequence of parameters $\lambda \in (\Lambda_1,\Lambda_2)$:

$$\bar{\lambda}_N^*, \bar{\lambda}_{N+1}, \bar{\lambda}_{N+1}^*, \bar{\lambda}_{N+2}, \bar{\lambda}_{N+2}^*, \cdots, \bar{\lambda}_{N+p}, \bar{\lambda}_{N+p}^*, \cdots,$$

to which belongs the sequence of functions:

$$\bar{y}_N^*, \bar{y}_{N+1}, \bar{y}_{N+1}^*, \bar{y}_{N+2}, \bar{y}_{N+2}^*, \cdots, \bar{y}_{N+p}, \bar{y}_{N+p}^*, \cdots,$$

so that $\bar{y}_{N+p} = y(x,\bar{\lambda}_{N+p})$, $\bar{y}^{*}_{N+p} = y(x,\bar{\lambda}^{*}_{N+p})$ are solutions of the differential equation (a), which fulfil the first two of the boundary conditions (10) and the condition $y'(c,\bar{\lambda}_{N+p}) = 0$, $y'(c,\bar{\lambda}^{*}_{N+p}) = 0$. Therefore $\bar{y}'_{N+p}$ and $\bar{y}^{*}_{N+p}$ have in (a, c) exactly N+p zero points.

If $\delta(\lambda) \equiv 0$, then the theorem is proved.

Assuming now that $\delta(\lambda) \neq 0$ for any $\lambda \in (\Lambda_1, \Lambda_2)$, then $\frac{(py')'(c,\lambda)}{y'(c,\lambda)}$ assumes for $\bar{\lambda} \in (\bar{\lambda}^{*}_{N+p}, \bar{\lambda}_{N+p+1})$ values from $+\infty$ to $-\infty$, because

$$\lim_{\lambda \to \bar{\lambda}^{*}_{N+p}+} \frac{(py')'(c,\lambda)}{y'(c,\lambda)} = +\infty, \qquad \lim_{\lambda \to \bar{\lambda}_{N+p+1}-} \frac{(py')'(c,\lambda)}{y'(c,\lambda)} = -\infty.$$

Consequently there exists a parameter $\lambda_{N+p} \in (\bar{\lambda}^{*}_{N+p}, \bar{\lambda}_{N+p+1})$ for which the relation

$$\frac{(py')'(c,\lambda_{N+p})}{y'(c,\lambda_{N+p})} = \frac{8(\lambda_{N+p})}{\delta(\lambda_{N+p})}$$

is valid; and hence $y'(x,\lambda_{N+p})$ has in (a,c) exactly N + p zero points. This complete the proof.

The other cases can be similarly proved.

We note here that the boundary value problem with the boundary conditions (1) is treated in a previous paper (Abdel Karim, 1976).

Further Types of Boundary Value Problems

In this paragraph further types of boundary value problems with analogical boundary conditions in three points will be investigated.

The following theorem is established:

Theorem 7

Let the hypotheses of theorem 6 hold. Theorem 6 is still true, if the first condition $y(a,\lambda) = 0$ in the boundary conditions (1) and (6)-(13) is replaced either by $y'(a,\lambda) = 0$, or by $(py')'(a,\lambda) = 0$.

Proof

Let us consider the boundary value problem *e.g.* with the boundary conditions (compare with (13))

$$
(14) \qquad \begin{aligned} &y'(a,\lambda)=0 \\ &\alpha(\lambda)\ y'(b,\lambda)-\beta(\lambda)\ (py')'(b,\lambda)=0 \\ &(\lambda)\ y'(c,\lambda)-\delta(\lambda)\ (py')'(c,\lambda)=0. \end{aligned}
$$

Let $y=c_1y_1+c_2y_3$ be the band of solutions of the second kind of the differential equation (a) at the point a, where y_1 and y_3 satisfy the first and third relations of (4) respectively. Then the first boundary condition of (14) is satisfied. Choose c_1 and c_2 such that the second conditionof (14) is satisfied. For this purpose it is enough to choose c_1 and c_2 such that the system

$$
\begin{aligned} &c_1y_1'(b,\lambda)+c_2y_3'(b,\lambda)=\beta(\lambda) \\ &c_1(py_1')'(b,\lambda)+c_2(py_3')'(b,\lambda)=\alpha(\lambda) \end{aligned}
$$

is satisfied. The coefficient determinant of this system is (see theorem 1)

$$
\begin{vmatrix} y_1'(b,\lambda) & y_3'(b,\lambda) \\ (py_1')'(b,\lambda) & (py_3')'(b,\lambda) \end{vmatrix} = w_2''(b,\lambda) \neq 0.
$$

Then the proof proceeds likewise to the proof of theorem 6, but we have here to use the oscillation theorem for the band of solutions of the second kind.

With the help of the band of solutions of the second and third kind, and the oscillation theorem for the considered band, the other types of boundary value problems can be analogously proved.

By virtue of this theorem, 18 types of boundary value problems in three points are solved.

Corollary

Setting for α, β, γ, δ special values in the preceding boundary conditions, we get different types of special boundary conditions, for example

$$y(a,\lambda)=y(b,\lambda)=y(c,\lambda)=0,$$

or

$$y'(a,\lambda)=y'(b,\lambda)=y'(c,\lambda)=0,$$

or

$$(py')'(a,\lambda)=(py')'(b,\lambda)=(py')'(c,\lambda)=0,$$

or

$$y(a,\lambda)=y'(a,\lambda)=(py')'(c,\lambda)=0,$$

or

$$(py')'(a,\lambda)=y'(b,\lambda)=y'(c,\lambda)=0$$

and so on.

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بعض مسائل القيمة الحدية من المرتبة الثالثة في ثلاث نقط

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في هذا البحث ندرس عدة أنواع من مسائل القيمة الحدية في ثلاث نقط لحلول المعادلة التفاضلية

(ق (س) صَ)″ + ك (س ، λ) ص = صفراً ،

حيث ق (س) > . ، ك (س ، λ) ≤ . دوال متصلة في س ∋ (− ∞ ، ∞) ، λ ∋ (١٨ ، ١.٢) ؛ والعلاقة ك (س ، λ) ≡ صفراً لا تسري في أي فترة .

وسوف نستخدم في حل مسائل القيمة الحدية خواص ما نسميه « حزم الحلول » من النوع الأول والثاني والثالث للمعادلة التفاضلية المذكورة ، وكذلك « النظرية التذبذبية » لحزم الحلول .