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Z_2-Graded Generalizations of Some Classical Lie Algebras and Curvature Structures

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By means of graded-symmetric and graded-skew bilinear-forms, Z_2-graded generalizations of the pseudo-orthogonal, symplectic and pseudo-unitary algebras of transformations on Z_2-graded real vector spaces can be defined. Suitable standard transformations in these ortho-symplectic and graded pseudo-unitary algebras are graded generalizations of SINGER and THORPE's riemannian curvature structures. In the finite-dimensional case they generate the corresponding algebras whence their graded commutation relations contain all information on those algebras. It turns out that the graded commutation relations of the trivial (pseudo-orthogonal) and the graded pseudo-skewhermitian (graded pseudo-orthogonal) curvatures are necessary and sufficient conditions for the graded Jacobi identity of the standard embedding algebra. As a special case a symplectic curvature concept results.

1. *Introduction to lie-graded algebras*

Let Δ be one of the commutative rings Z or Z_2, the ground-field K be R or C, and the K-vector space V be graded of type Δ, *i.e.* $V = \bigoplus_{i \in \Delta} V_i$ (direct sums). Let $[,]_\pm : V \times V \rightarrow V$ be a graded algebra composition, *i.e.* $[V_k, V_1]_\pm \subset V_{k+1}$ for all k, 1 in Δ .

We call the pair $(V,[,]_\pm)$ a Δ-lie graded algebra if in addition

(LGA.1) $[x_k, y_1]_\pm = -(-1)^{k1}[y_1, x_k]_\pm$ (*graded antisymmetry*)

(LGA.2) $[[x_k, y_1]_\pm, a]_\pm = [x_k, [y_1, a]_\pm]_\pm - (-1)^{k1}[y_1, [x_k, a]_\pm]_\pm$

(*graded Jacobi identity*),

for all x_k in V_k, y_1 in V_1 and a in V arbitrary.

These algebra should not be confused with lie algebras which admit a compatible graduation. First they were studied by Gerstenhaber (1973), Nyenhuis and Richardson (1964), recently by Djokovic (1976), Freund and Kaplansky (1976), Pais and Rittenberg (1975), and the author (1977 a, b and c). In 1974 Haefliger used them for the cohomology of vector fields. For generalizations of the graduation Δ and the *commutation factor* $(-1)^{kl}$ see Bourbaki (1974) chap. III, 10 sections 1,4,6. Berezin and Kac (1971) studied a generalized lie group concept the local tangent structure of which is a Lie-graded algebra.

A (Δ-*lie-graded*) *subalgebra* is a graded subspace $U = \bigoplus_i U_i$ with $U_i \subset V_i$ and $[U_k, U_1]_\pm \subset U_{k+1}$, a ($\Delta$-*lie-graded*) *ideal* is such a subalgebra with $[U_k, V_1]_\pm \subset U_{k+1}$. A *homomorphism* is a homogeneous (necessarily of degree 0) linear mapping φ, *i.e.* $\varphi V_k \subset V'_k$, which is a homomorphism of the compositions on V and V'. It is straightforeward to prove that ideals are exactly the kernels of homomorphisms, and that the class of Δ -lie-graded algebras is a category the morphisms being the homomorphisms.

The standard example of a lie-graded algebra is a graded associative algebra supplied with the *graded commutator* $2[x_k, y_1]_\pm = x_k y_1 - (-1)^{k1} y_1 x_k$. For instance the algebra end $^\pm V = \bigoplus_{i \in \Delta} \text{end}_i V$, where $\text{end}_i V$ is the subspace of endomorphisms *of degree* i of the graded vector space V, *i.e.* $\text{end}_i V(V_k) \subset V_{k+i}$, is of this type. If V is finite dimensional, end $^\pm V$ = end V (Bourbaki 1974) remark in chap. II 11.6 Any subspace of end $^\pm V$ closed under graded commutation again is a Lie-graded algebra. A *representation* is a homomorphism into some end $^\pm V$. Now Bourbaki (1979) chap. III 10.2 defines generallized Δ – graded derivations which according to prop. 1 in 10.4 span Δ-lie-graded algebras. There are two important special cases of such graded derivations in end $^\pm V$: (i) Given a graded K-algebra (V,.) the spaces $\text{der}_i(V,.)$ of *graded derivations of degree* i, $D^{(i)} \in \text{end}_i V$, *i.e.* $D^{(i)}(V_k) \subset V_{i+k}$ and (1) $D^{(i)}(x_k.a) = (D^{(i)}x_k).\, a + (-1)^{ik} x_k.\, D^{(i)} a$, $x_k \in V_k$, $a \in V$, sum up to a Δ-lie-graded subalgebra $\text{der}^\pm(V,.) = \bigoplus_{i \in \Delta} \text{der}_i(V,.)$ of end$^\pm V$; clearly $\text{der}^\pm(V,.)$ is identical with end $^\pm V$ if. reduces to the trivial zero-composition. (ii) Given a bilinear form $<\,,\,>$ on the Δ -graded vector space V, $D^{(i)} \in \text{end}_i V$ is said to be a *graded derivation of degree* i of (V, $<\,,\,>$) if for x_k in V_k and any a in V

(2) $<D^{(i)}x_k, a> + (-1)^{ik} <x_k, D^{(i)}a> = 0;$

the spaces $\text{der}_i(V, <,>)$ of such graded derivations sum up to a Δ-lie-graded subalgebra $\text{der}^\pm(V, <,>)$ of end$^\pm V$, which again is identical to the latter if $<,>$ is the zero-bilinear form. In the following sections case (ii) is used to describe a class of Z_2-graded generalizations of some classical simple real lie algebras. Another class is studied in (Pais

& Rittenberg, 1975). Algebras of class (i) might be interesting as well as for physical applications in the classification of elementary particles, (V,.) then being a graded generalization of a Jordan algebra of observables.

(LGA.2) means that the *left multiplication* ad in a Lie-graded algebra $(V,[,]_{\pm})$, defined by $\mathrm{ad}(x_k)a=[x_k,a]_{\pm}$, is a representation into the lie-graded algebra $\mathrm{der}^{\pm}(V,[,]_{\pm})$, called the *adjoint* representation.

A bilinear form $<,>$ on the Δ-graded vector space V will be said to be *graded symmetric* if $<x_k,y_l>=(-1)^{kl}<x_k,y_l>$, resp. *graded skew* if $<x_k,y_l>=-(-1)^{kl}<y_l,x_k>$. In the following $<,>$ always denotes a graded symmetric, $\nless,\ngtr$ a graded skew bilinear form, and only the case $\Delta=Z_2$ is considered. Hence $V=V_0\oplus V_1$. Given $(V,<,>)$ or $(V,\nless,\ngtr)$, the restrictions τ_0 resp. σ_1 of $<,>$ to V_0 resp. V_1 then are symmetric resp. skew, the restriction σ_0 resp. τ_1 of $\nless$, $\ngtr$ to V_0 resp. V_1 are skew resp. symmetric bilinear forms, *i.e.* (V_0,τ_0) and (V_1,τ_1) are pseudo-orthogonal, (V_0,σ_0) and (V_1,σ_1) are symplectic vector spaces if the bilinear forms are non-degenerate. Moreover the decomposition $V=V_0\oplus V_1$ will be assumed to be $<,>$ – resp. $\nless,\ngtr$ -orthogonal, *i.e.* $<x_k,y_l>=\nless x_k,y_l\ngtr=0$ if $k\neq l$.

Throughout the following x_k will be in V_k, y_l in V_l, z_m in V_m, w_r in V_r and a in V arbitrary.

2. *General and special linear, pseudo-orthogonal and symplectic Lie-graded algebras of graduation type* Z_2

The *general linear* algebra $gl^{\pm}(V,K)$ is given by $\mathrm{end}^{\pm}V$ and the graded commutator. A typical element in $\mathrm{end}_{k+l}V$ is given by

(3) $$G(x_k,y_l)a=<y_l,a>x_k.$$

A verification gives the *graded commutation realations*

(4) $$[G(x_k,y_l),G(z_m,w_r)]_{\pm}=<y_l,z_m>G(x_k,w_r)-(-1)^{(k+l)(m+r)}<w_r,x_k>G(z_m,y_l).$$

If V is finite dimensional and $<,>$ non-degenerate, the G(,) generate $\mathrm{end}^{\pm}V$ linearly and (4) may be called the *graded commutation relations* of $gl^{\pm}(V,K)$. Then obviously the trace of $G(x_k,y_l)$ is $<y_l,x_k>$. Hence

(5) $$S(x_k,y_l)=G(x_k,y_l)-(\dim V)^{-1}<y_l,x_k>\mathrm{id}_V$$

is traceless and of degree $k+l$:

$$S(x_0,y_0)a_o=\tau(y_0,a_0)x_0-(\dim V)^{-1}\tau(y_0,x_0)a_0$$

$$
(6)\quad
\begin{aligned}
&S(x_1,y_1)a_0 = -(\dim V)^{-1}\sigma(y_1,x_1)a_0\\
&S(x_0,y_1)a_0 = 0 \quad S(x_1,y_0)a_0 = \tau(x_0,a_0)x_1\\
&S(x_0,y_0)a_1 = -(\dim V)^{-1}\tau(y_0,x_0)a_1\\
&S(x_1,y_1)a_1 = \sigma(y_1,a_1)x_1 - (\dim V)^{-1}\sigma(y_1,x_1)a_1\\
&S(x_0,y_1)a_1 = \sigma(y_1,a_1)x_0 \qquad S(x_1,y_0)a_1 = 0
\end{aligned}
$$

Dropping the terms with dim V the corresponding expressions for the G(,) result. A simple calculation gives the same graded commutation relations (4) for the S(,), explicitely

$$(4a)\qquad [S(x_0,y_0),S(z_0,w_0)]_- = \tau(y_0,z_0)S(x_0,w_0) - \tau(w_0,x_0)S(z_0,y_0)$$

$$(4b)\qquad [S(x_1,y_1),S(z_1,w_1)]_- = \sigma(y_1,z_1)S(x_1,w_1) + \sigma(w_1,x_1)S(z_1,y_1)$$

$$(4c)\qquad [S(x_0,y_0),S(z_1,w_1)]_- = 0$$

$$
(4d)\quad
\begin{aligned}
&[S(x_0,y_1),S(z_0,w_1)]_+ = [S(x_1,y_0),S(z_1,w_0)]_+ = 0\\
&[S(x_0,y_1),S(z_1,w_0)]_+ = \sigma(y_1,z_1)S(x_0,w_0) - \tau(w_0,x_0)S(z_1,y_1)
\end{aligned}
$$

$$
(4e)\quad
\begin{aligned}
&[S(x_0,y_0),S(z_0,w_1)]_- = \tau(y_0,z_0)S(x_0,w_1)\\
&[S(x_0,y_0),S(z_1,w_0)]_- = -\tau(w_0,x_0)S(z_1,y_0)
\end{aligned}
$$

$$
(4f)\quad
\begin{aligned}
&[S(x_1,y_1),S(z_0,w_1)]_- = -\sigma(w_1,x_1)S(z_0,w_1)\\
&[S(x_1,y_1),S(z_1,w_0)]_- = \sigma(y_1,z_1)S(x_1,w_0)
\end{aligned}
$$

(4) are the graded commutation relations of $gl^{\pm}(V,K)$ resp. $sl^{\pm}(V,K)$. (4a) - (4e) show that the zero-components are direct lie algebra sums of the corresponding classical lie algebras. All this can be given as well in terms of any non-degenerate bilinear form on V for which $V_0 \oplus V_1$ is an orthogonal sum.

To get the Z_2-*lie-graded pseudo-orthogonal* algebra $der^{\pm}(V,<,>)$ define

$$(7)\qquad R(x_k,y_1)a = <y_1,a>x_k - (-1)^{k1} <x_k,a>y_1$$

with $\quad R(y_1,x_k) = -(-1)^{k1}R(x_k,y_1)$, explicitely

(7a) $R(x_0,y_0)a_0=\tau_0(y_0,a_0)x_0-\tau_0(x_0,a_0)y_0 \quad R(x_1,y_1)a_0=0$

(7b) $R(x_0,y_1)a_0=-\tau_0(x_0,a_0)y_1$

(7c) $R(x_0,y_0)a_1=0 \quad R(x_1,y_1)a_1=\sigma_1(y_1,a_1)x_1+\sigma_1(x_1,z_1)y_1$

(7d) $R(x_0,y_1)a_1=\sigma(y_1,a_1)x_0.$

This shows that $R(x_k,y_1)$ is in $\mathrm{end}_{k+1}V$. A tedious but straightforeward verification gives

(8) $[R(x_k,y_1),R(z_m,w_r)]_{\pm}=<y_1,z_m>R(x_k,w_r)-(-1)^{k1}<x_k,z_m>R(y_1,w_r)-(-1)^{mr}$ $<y_1,w_r>R(x_k,z_m)\mp(-1)^{k1}(-1)^{mr}<x_k,w_r>R(y_1,z_m).$

The following inspection of the various special choices of the indices shows that these are graded commutation ralations of a lie-graded sub-algebra of $gl^{\pm}(V,K)$:

(8a) $[R(x_0,y_0),R(z_0,w_0)]_-=\tau_0(y_0,z_0)R(x_0,w_0)-\tau_0(x_0,z_0)R(y_0,w_0)$ $-\tau_0(y_0,w_0)R(x_0,z_0)+\tau_0(x_0,w_0)R(y_0,z_0)$

together with the first equation (7a) gives the wellknown commutation relations of the (finite-dimensional) pseudo-orthogonal lie algebra der $(V_0,\tau_0)=\{A\in\mathrm{end}V_0/\tau_0(Ax_0,y_0)+\tau_0(x_0,Ay_0)=0\}$, (Jacobson, 1966), p. 232.

(8b) $[R(x_1,y_1),R(z_1,w_1)]_-=\sigma_1(y_1,z_1)R(x_1,w_1)+\sigma_1(x_1,z_1)R(y_1,w_1)$ $+\sigma_1(y_1,w_1)R(x_1,z_1)+\sigma_1(x_1,w_1)R(y_1,z_1)$

together with the second equation in (7c) gives the commutation relations of the symplectic algebra der $(V_1,\sigma_1)=\{D\in\mathrm{end}V_1/\sigma_1(Dx_1,y_1)+\sigma_1(x_1,Dy_1)=0$ for all $x_1,y_1\in V_1\}$.

(8c) $[R(x_0,y_0),\ R(z_1,\ w_1)]\ -=0$

together with (a) and (b) shows that the $R(x_0,y_0)$ and $R(x_1,y_1)$ span the direct lie algebra sum of the pseudo-orthogonal lie algebra on (V_0,τ_0) and the sympletic algebra on (V_1,σ_1). In addition

(8d) $[R(x_0,y_1),R(z_0,w_1)]_+=-\tau_0(x_0,z_0)R(y_1,w_1)-\sigma_1(y_1,w_1)R(x_0,z_0)$

(8e) $[R(x_0,y_0),R(z_0,w_1)]_-=\tau_0(y_0,z_0)R(x_0,w_1)-\tau_0(x_0,z_0)R(y_0,w_1)$

(8f) $[R(x_1,y_1),R(z_0,w_1)]_-=-\sigma_1(y_1,w_1)R(x_1,z_0)-\sigma_1(x_1,w_1)R(y_1,z_0).$

A verification shows that $R(x_k,y_1) \in \mathrm{der}_{k+1}(V,<,>)$, *i.e.* that (2) holds for $R(x_k,y_1) = D^{(k+1)}$. If V is finite-dimensional and $<,>$ non-degenerate dimensional arguments show that the R(,) span $\mathrm{der}^{\pm}(V,<,>)$. Then the trace of R(,) vanishes. Hence the equations (8) are the graded commutation relations of the Z_2-graded *ortho-symplectic* algebra, described also in (Pais and Rittenberg, 1975).

If V is finite-dimensional over $K = R$ there is a natural basis in which the matrix of $<,>$ is $I_{<,>} = \mathrm{diag}(I_\tau, J_\sigma)$ where $I_\tau = \mathrm{diag}(\mathrm{id}_p, -\mathrm{id}_q)$ with $p+q = n_0 = \dim V_0$ and $J_\sigma = \mathrm{antidiag}(-\mathrm{id}_d, \mathrm{id}_d)$ (if σ is non-degenerate which implies $2d = n_1 = \dim V_1$). The matrix of $D^{(i)}$ in (2) then is

$$\text{(9)} \qquad \begin{pmatrix} A & B \\ -J_\sigma B^t I_\tau & D \end{pmatrix} \text{ with } A^t I_\tau + I_\tau A = 0 \text{ and } D^t J_\sigma + J_\sigma D = 0,$$

where A is a square n_0 matrix, D a square n_1 matrix and B an arbitrary rectangular matrix of n_0 rows and n_1 columns. The dimension of the real ortho-symplectic algebra is $\frac{1}{2}n(n+1) - n_0$ with $n = n_0 + n_1 = \dim V$. Since the concept of a graded (orthogonal) curvature structure, to be discussed in the last section, has a graded-symplectic analog, we add the corresponding facts on the Z_2-graded symplectic algebra $\mathrm{der}^{\pm}(V, \nless, \ngtr)$ although it results from $\mathrm{der}^{\pm}(V,<,>)$ by interchanging the indices 0 and 1, *i.e.* by taking the derived graduation of Δ by means of the nontrivial automorphism of Z_2. described in example (2) chap. II 11.1 of (Bourbaki, 1974). The typical linear transformation in $\mathrm{der}_{k+1}(V, \nless, \ngtr)$ is defined by

$$P(x_k,y_1)a = \nless y_1, a \ngtr x_k + (-1)^{k1} \nless x_k, a \ngtr y_1$$

with $P(y_1,x_k) = (-1)^{k1} P(x_k,y_1)$ and vanishing trace. Their graded commutation relations are (Tilgner, 1977 a).

$$[P(x_k,y_1), P(z_m,w_r)]_{\pm} = \nless y_1, z_m \ngtr P(x_k,w_r) + (-1)^{k1} \nless x_k, z_m \ngtr P(y_1,w_r) + (-1)^{mr} \nless y_1, w_r \ngtr P(x_k,z_m) + (-1)^{k1}(-1)^{mr} \nless x_k, w_r \ngtr P(y_1,z_m).$$

Denoting the matrix of $\nless, \ngtr$ by $I_{\nless\ngtr} = \mathrm{diag}(J_\sigma, I_\tau)$, the typical matrix $D^{(i)}$ in (2) now is $\begin{pmatrix} D & J_\sigma B^t I_\tau \\ B & A \end{pmatrix}$ with the same conditions on A,B,D as in (9).

3. *Z_2-graded pseudo-unitary algebras*

Let us first describe the ordinary *speudo-unitary* Lie algebra u(p,q) on a finite-dimensional complex vector space U: It is the space of complex square $p+q$ matrices M leaving invariant the nondegenerate hermitian form $<<x,y>> = x^t I_\tau y^*$, where $I_\tau = \mathrm{diag}(\mathrm{id}_p, -\mathrm{id}_q)$, which means $<<Mx,y>> + <<x,My>> = 0$ for all $x,y \in U$ or in matrix form $M^t I_\tau + I_\tau M = 0$. $M = A + iB \in u(p,q)$ is equivalent to $A^t I_\tau + I_\tau A = 0$ and $B^t I_\tau$

$=I_\tau B$. Consider on the real 2n-dimensional vector space V with $n=p+q$, a skew bilinear form σ and a symmetric one $\hat{\tau}$ with matrices $\begin{pmatrix} 0 & -I_\tau \\ I_\tau & 0 \end{pmatrix}$ and $\begin{pmatrix} I_\tau & 0 \\ 0 & I_\tau \end{pmatrix}$ resp.. It is well known that

$$(10)\qquad M=A+iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} =: \hat{M}$$

is a Lie algebra isomorphism of u(p,q) onto the space of real square 2n matrices $\hat{M}$ subject to $\hat{M}\,'I_\tau + I_\tau \hat{M} = 0 = \hat{M}\,'J_\sigma + J_\sigma \hat{M}$, *i.e.* onto the intersection of so (2p,2p;R) with the symplectic algebra on (V,σ)(the latter being conjugate but not necessarily equal to sp(2n,R)). This real version of u(p,q) will be denoted by $u_r(p,q)$ in the following. Now the matrix $J=\begin{pmatrix} 0 & -id_n \\ id_n & 0 \end{pmatrix}$ with $J^2=-id_{2n}$ is a complex structure on V such that given two of the three structures $\hat{\tau}$, σ, J, the third is determined uniquely, explicitely

$$(11)\qquad \hat{\tau}(Jx,z) = -\sigma(x,z) \qquad \sigma(Jx,z) = \hat{\tau}(x,z)$$

$$\hat{\tau}(x,Jz) = \sigma(x,z) \qquad \sigma(x,Jz) = -\hat{\tau}(x,z)$$

J defines a Cartan decomposition of the symplectic algebra on (V,σ) into the two eigenspaces of eigenvalues 1 and -1 of the involutive automorphism $\hat{M} \mapsto J\hat{M}J^{-1}$ of the form $2\hat{M} = \hat{M} + J\hat{M}J^{-1} \oplus \hat{M} - J\hat{M}J^{-1}$ where the first eigenspace of eigenvalue 1 is $u_r(p,q)$. The element R(x,y) of so (2p,2q;R) defined in the first equation in (7a) hence can be used to define a typical element U(x,y) of $u_r(p,q)$ by

$$(12)\qquad U(x,y)a = \{R(x,y)+JR(x,y)J^{-1}\}a = \hat{\tau}(y,a)x - \hat{\tau}(x,a)y + \sigma(a,y)Jx - \sigma(a,x)Jy$$

with $U(y,x) = -U(x,y)$ and the commutation relations

$$(13)\qquad \begin{aligned}[U(x,y),U(z,w)] &= \{\hat{\tau}(y,z)U(x,w) + \sigma(z,y)U(Jx,w)\} - \{\hat{\tau}(x,z)U(y,w) \\ &+ \sigma(z,x)U(Jy,w)\} - \{\hat{\tau}(y,w)U(x,z) + \sigma(w,y)U(Jx,z)\} + \{\hat{\tau}(x,w)U(y,z) \\ &+ \sigma(w,x)U(Jy,z)\}\end{aligned}$$

If V is finite-dimensional the U(x,y) span $u_r(p,q)$.

To get a Z_2-graded generalization of $u_r(p,q)$ on $V=V_0 \oplus V_1$, we introduce besides $<,>$ and $\langle\!\langle,\rangle\!\rangle$ a complex structure $J \in end_0 V$, *i.e.* $J^2=-id$, which is the diagonal of two complex structures J_0 on V_0, J_1 on V_1 related to $\hat{\tau}_0$ and σ_0, $\hat{\tau}_1$ and σ_1 as indicated in(11).

Hence

$$\nless x_k, Jz_m \ngtr = -(-1)^{km} < x_k, z_m > = - < z_m, x_k >$$

$$\nless Jx_k, z_m \ngtr = (-1)^{km} < x_k, z_m > = < z_m, x_k >$$

(14)

$$< x_k, Jz_m > = (-1)^{km} \nless x_k, z_m \ngtr = - \nless z_m, x_k \ngtr$$

$$< Jx_k, z_m > = -(-1)^{km} \nless x_k, z_m \ngtr = \nless z_m, x_k \ngtr$$

The Z_2-*Lie-graded pseudo-unitary algebra* $u_r^{\pm}(V, <,>, J)$ is now defined as $\mathrm{der}^{\pm}(V, <, >) \cap \mathrm{der}^{\pm}(V, \nless, \ngtr)$. Again a Cartan decomposition of $\mathrm{der}^{\pm}(V, \nless, \ngtr)$ can be used to construct its standard linear transformation: J induces an involutive automorphim of $\mathrm{der}^{\pm}(V, \nless, \ngtr)$ by

(15) $$J: A^{(i)} \mapsto (-1)^i J A^{(i)} J^{-1}, \qquad A^{(i)} \in \mathrm{der}_i(V, \nless, \ngtr),$$

and $u_r^{\pm}(V, <,>, J)$ is exactly the eigenspace of eigenvalue 1. Hence

(16) $$U(x_k, y_1) = R(x_k, y_1) + (-1)^{k+1} J R(x_k, y_1) J^{-1},$$

$$U(x_k, y_1)a = \nless y_1, a \ngtr x_k - (-1)^{k1} \nless x_k, a \ngtr y_1$$

$$+ (-1)^{k+1} \nless a, y_1 \ngtr Jx_k - (-1)^{k+1} (-1)^{kl} \nless a, x_k \ngtr Jy_1.$$

with $U(y_1, x_k) = -(-1)^{k1} U(x_k, y_1)$ is in $u_r^{\pm}(V, \nless, \ngtr, J)$. Using the $<,>$- and $\nless, \ngtr$-orthogonality of $V_0 \oplus V_1$ a simple but tedious calculation gives the graded commutation relations

(17) $$[U(x_k, y_1), U(z_m, w_r)]_{\pm} = \{ < y_1, z_m > U(x_k, w_r) + (-1)^{k+1} \nless z_m, y_1 \ngtr U(Jx_k, w_r) \} - (-1)^{k1} \{ < x_k, z_m > U(y_1, w_r) + (-1)^{k+1} \nless z_m, x_k \ngtr U(Jy_1, w_r) \} - (-1)^{mr} \{ < y_1, w_r > U(x_k, z_m) + (-1)^{k+1} \nless w_r, y_1 \ngtr U(Jx_k, z_m) \} + (-1)^{k1}(-1)^{mr} \{ < x_k, w_r > U(y_1, z_m) + (-1)^{k+1} \nless w_r, x_k \ngtr U(Jy_1, z_m) \}.$$

If V is finite-dimentional the $U(x_k, y_1)$ span $u_r^{\pm}(V, <,>, J)$. To verify that (17) are graded commutation relations of a graded algebra the various special choices of the indices must be discussed: The 0-0-case is (13), which together with

$$[U(x_1, y_1), U(z_1, w_1)]_- = \sigma_1(y_1, z_1) U(x_1, w_1) + \hat{\tau}_1(y_1, z_1) U(J_1 x_1, w_1)$$

(17b) $$- \{ \sigma_1(x_1, z_1) U(y_1, w_1) + \hat{\tau}_1(x_1, z_1) U(J_1 y_1, w_1) \}$$

$$- \{ \sigma_1(y_1, w_1) U(x_1, z_1) + \hat{\tau}(y_1, w_1) U(J_1 x_1, z_1) \}$$

$$+ \sigma_1(x_1, w_1) U(y_1, z_1) + \hat{\tau}_1(x_1, w_1) U(J_1 y_1, z_1),$$

(17c) $[U(x_0,y_0),U(z_1,w_1)]_- = 0$

shows that the 0-component of this algebra is the direct lie algebra sum of the pseudo-unitary lie algebra $(V_0,\hat{\tau}_0,\sigma_0)$ and (as will be shown below) that on $(V_1,\hat{\tau}_1,\sigma_1)$. The anticommutation relations are

(17d) $$[U(x_0,y_1),U(z_0,w_1)]_+ = -\hat{\tau}_0(x_0,z_0)U(y_1,w_1) - \sigma_0(x_0,z_0)U(J_1\, y_1,w_1) - \sigma_1(y_1,w_1)U(x_0,z_0) + \hat{\tau}_1(y_1,w_1)U(J_0\, x_0,z_0),$$

and the two remaining commutation relations (e) and (f) are of similar type such that $[U(x_0,y_0),U(z_0,w_1)]_-$ and $[U(x_1,y_1), U(z_0,w_1)]-$ are linear combinations of the $U(a_o,b_1)$, *i.e.* in the 1-component of the algebra. It remains to identify the space spanned by the $U(x_1,y_1)$ in the o-component as the pseudo-unitary Lie algebra on V_1: $U(x_1,y_1)$ in (16) and (17b) is the pseudo-unitary standard transformation $U(Jx,y)$ in (12) and an easy calculation gives from (17b) the commulation relations (13) for the $U(J_1x_1,y_1)$.

In a basis of a finite-dimensional V in which $\hat{\tau}_i$ has the matrix $I_{\hat{\tau}_i}$ and σ_i the matrix J_{σ_i}, the elements of a matrix form of $u_{\hat{\tau}}^{\pm}(V,<,>,J)$ are

(18) $$\begin{pmatrix} A & B \\ -J_{\sigma 1}B'I_{\hat{\tau}} & D \end{pmatrix}$$

where A is a square $2n_0$ matrix subject to $A'I_{\hat{\tau}_0} + I_{\hat{\tau}_0}A = 0 = A'J_{\sigma_0} + J_{\sigma_0}A$, D a square $2n_1$ matrix subject to $D'I_{\tau_1} + I_{\tau_1}D = 0 = D'J_{\sigma_1} + J_{\sigma_1}D$, and B a rectangular matrix with $2n_0$ rows and $2n_1$ columns subject to the equation $J_{\sigma_0}\, BI_{\tau_1} = -I_{\hat{\tau}0}BJ_{\sigma 1}$. This leaves only $2n_0n_1$ matrix elements of B independent. Hence the real dimension of the graded pseudo-unitary algebra is $(n_0+n_1)^2$. The involutive automorphism (15) of $\mathrm{der}^{\pm}(V, \nless, \ngtr)$ is in matrix form

$$\hat{J}:\begin{pmatrix} D & J_{\sigma 0}B'I_{\hat{\tau}1} \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} J_0DJ_0^{-1} & -J_0J_{\sigma 1}B'I_{\hat{\tau}1}J_1^{-1} \\ -J_1BJ_0^{-1} & J_1AJ_1^{-1} \end{pmatrix}$$

It remains to generalize some well known embeddings of classical Lie algebras, for instance su(p,q), gl(n,K) in sp(2n,K) and so(p,q;R) in $u_r(p,q)$.

4. *Z_2-graded curvature structure*

Let V be finite-dimensional, <,> a graded-symmetric bilinear form as above. A Z_2-

graded curvature structure on $(V, <,>)$ is defined as the linear continuation of a bilinear mapping

$$C: V_k \times V_1 \to \mathrm{end}_{k+1} V$$

subject to the axioms

(GC.1) $\quad C(y_1, x_k) = -(-1)^{k1} C(x_k, y_1)$

(GC.2) $\quad (-1)^{km} C(x_k, y_1) z_m + (-1)^{1m} C(z_m, x_k) y_1 + (-1)^{k1} C(y_1, z_m) x_k = 0$

(*graded Bianchi identity*)

(GC.3) $\quad < C(x_k, y_1) z_m, w_r > + (-1)^{(k+1)m} < z_m, C(x_k, y_1) w_r > = 0.$

(GC.3) obviously means $C(x_k, y_1) \in \mathrm{der}_{k+1}(V, <,>)$. C induces a linear mapping of degree zero of $V \oplus V$ with its total graduation Z_2, (Bourbaki, 1974), remark in chap. III 11.5, into $\mathrm{end}^{\pm} V$. The *trivial* curvature structure on $(V, <,>)$ is R defined in (7). Denoting the left hand side of (GC.2) by

$\sum(x_k, y_1, z_m)$ we have $0 = (-1)^{mr} < \sum(x_k, y_1, z_m), w_r > -$

$(-1)^{k(m+r)} < \sum(y_1, w_r, x_k), z_m > - (-1)^{k1+1m+kr} < \sum(w_r, z_m, y_1), x_k > +$

$(-1)^{1(m+r)} < \sum(z_m, x_k, w_r), y_1 > = (-1)^{(k+r)m} < C(x_k, y_1) z_m, w_r > -$

$(-1)^{km} < C(x_k, y_1) w_r, z_m > - (-1)^{k1+kr+1m+1r} < C(w_r, z_m) y_1, x_k > +$

$(-1)^{kr+1r+1m} < C(w_r, z_m) x_k, y_1 >$, hence

(GC.4) $\quad < C(x_k, y_1) z_m, w_r > = (-1)^{(k+1)(m+r)} < C(z_m, w_r) x_k, y_1 >.$

This equation shows that C is a Z_2-graded generalization of Singer and Thorpe's riemannian curvature structure studied by (Kowalski, 1973; Kulkarni, 1968 and 1970; Nomizu, 1972) and in a little different notation by (Gray, 1971; Marcus, 1975 chap. 4, and Singer and Thorpe 1968). Let $\mathrm{curv}(V, <,>)$ denote the linear space spanned by the curvature structures on $V(<,>)$. It remains to generalize Singer and Thorpe's direct decomposition given by (Nomizu, 1972 and Singer and Thorpe, 1968) to $\mathrm{curv}(V, <,>)$. Given $C \in \mathrm{curv}(V, <,>)$ we call

(19) $\quad [A^{(i)}, C(z_m, w_r)]_{\pm} = C(A^{(i)} z_m, w_r) - (-1)^{mr} C(A^{(i)} w_r, z_m)$

Cartan's condition for $A^{(i)} \in der_i(V,<,>)$. If it is satisfied for all $C(x_k,y_1)=A^{(i)}$, *i.e.* $i=k+1$, then the image of C is a subalgebra of $der^+(V,<,>)$, if it is satisfied for all $A^{(i)} \in der_i(V,<,>)$ then this image even is an ideal. Choosing $A^{(i)}=C(x_k,y_1)$ and $C=R$ Cartan's condition reduces to the graded commutation relations (8) of the ortho-symplectic algebra.

The standard transformations $U(x_k,y_1)$ of $u^{\pm}_{i}(V,<,>,J)$ are J-dependent curvature structures on $(V,<,>)$ which may be called *J-pseudo-skewhermitian.* Again Cartan's condition for such a U reduces to the graded commutation relations (17).

The following is a graded generalization of a result due to E. Cartan, described for instance in (Helgason 1962) chap. IV. Given $C \in curv(V,<,>)$ we define a C-dependent graded skew algebra composition on $\bigoplus_{i\in\Delta}(C(V_k,V_1) \oplus V_i)$, $i=k+1$, by

(20) $$[A^{(k)}\oplus x_k, B^{(1)}\oplus y_1]_{\pm} = \tfrac{1}{2}(A^{(k)}B^{(1)}-(-1)^{k1}B^{(1)}A^{(k)}) - C(x_k,y_1)\oplus A^{(k)}y_1 - (-1)^{k1}B^{(1)}x_k$$

and linear continuation.

(21) *Lemma:* (20) is a lie-graded algebra composition if and only if (19) holds for all $A^{(i)}=C(x_k,y_1)$ and $i=k+1$. This algebra is called the *standard embedding* of C. Lemma (21) gives two lie-graded algebras: $\bigoplus_{0,1}(der_i(V,<,>)\oplus V_i)$ in the ortho-symplectic and $\bigoplus_{0,1}(u^{\pm}_{i}(V,<,>,J)_i\oplus V_i)$ in the pseudo-unitary case. The dimension of the latter obviously is $n(n+1)$. The linear mapping $\omega{:}A^{(i)}+v_i|\rightarrow A^{(i)}-v_i$ defines an involutive automorphism of these algebras whose eigenspace of eigenvalue $\pm$ is exactly the lie-graded subalgebra $\bigoplus_{i=k+1} C(V_k,V_1)$ resp. the subspace V. One verifies that the eigenspace of eigenvalue -1 is closed with respect to the graded double commutator, *i.e.* $[[V_k,V_1]_{\pm},V_m]_{\pm} \subset V_{k+l+m}$. Indeed there is a graded generalization of Lie triples, which are exactly of this type, such that the well know relation with Lie algebras remains valid. Cartan's condition for C then becomes one of the three axioms of a Z_2-graded generalized lie triple, written in terms of the left multiplication $C(x,y)a - [x,y,a]_{\pm}$ (see Tilgner, 1977b).

It remains to generalize the results on riemennian curvature, if possible including torsion, to the Z_2-graded case, and especially to the induced symplectic curvature on (V_1,σ_1).

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تعميات مـن درجـة ع٢ لبعض جـبر «لي» الـكلاسيكي وبني التقوس

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بواسطة الصيغ الثنائية الخطية المتماثلة الدرجة وشبه المتماثلة الدرجة يمكن أن نعرف تعميات من درجة ع٢ لجبر التحويلات شبه المتعامدة والسمبلكتك وشبه الواحدية على فضاءات المتجه الحقيقي من درجة ع٢ . والتحويلات المعيارية المناسبة في هـذا الجبر السمبلكتك العمودي وشبه الواحدي المدرج هي تعميات مدرجة لبُني تقوس ريمان وصيغتي « سنجر » و« ثورب » . وفي حالة البعد المنتهـي فـإنها تـولد الجـبر المقابل عندما تكون علاقات التبديل المدرج حـاوية لـكل المعلـومات عـن ذلك الجبر ، ويتبين أن علاقات التبديل المدرجة للتقـوسات التـافهة الشـبه متعـامدة والمدرجة بصورة شبه هرميتية هي شروط لازمة وكافية للجـبر المعيـاري مـن أجـل متطابقة « جاكوبي » المدرجة . وكحالة خاصة ينتج مفهوم التقوس السمبلكتك .