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On The Solutions of Quasilinear Differential Equations of The Fifth Order

Rahmi Ibrahim Ibrahim Abdel Karim
Department of Mathematics, Faculty of Science, Riyad University, Riyad, Saudi Arabia.

In this paper we study the fundamental properties of the solutions of certain quasilinear differential equations of the fifth order. We give the sufficient conditions for the solutions of these differential equations to be without zero points. Furthermore, a comparison theorem is derived, and a problem concerning the distribution of zeros of the solutions of these equations is investigated.

1. Introduction

In this paper we consider the differential equations of the fifth order of the form

(a) $$(p(x)y'')''' + q(x)y = 0$$

and

(b) $$(p(x)z''')'' - q(x)z = 0 ,$$

where $p(x) > 0$ and $q(x)$ are continuous functions of $x \in (-\infty, \infty)$.

The solutions $y_r(x)$ or $z_r(x)$ $(r = 1,2,3,4,5)$ of the differential equations (a) or (b) are linearly independent and form a fundamental system of solutions, if the determinant (Wronskian)

$$W(y_1,y_2,y_3,y_4,y_5)=\begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1' & y_2' & y_3' & y_4' & y_5' \\ y_1'' & y_2'' & y_3'' & y_4'' & y_5'' \\ (py_1'')' & (py_2'')' & (py_3'')' & (py_4'')' & (py_5'')' \\ (py_1'')'' & (py_2'')'' & (py_3'')'' & (py_4'')'' & (py_5'')'' \end{vmatrix}$$

or

$$W(z_1,z_2,z_3,z_4,z_5)=\begin{vmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1' & z_2' & z_3' & z_4' & z_5' \\ z_1'' & z_2'' & z_3'' & z_4'' & z_5'' \\ z_1''' & z_2''' & z_3''' & z_4''' & z_5''' \\ (pz''')' & (pz_2''')' & (pz_3''')' & (pz_4''')' & (pz_5''')' \end{vmatrix}$$

respectively is different from zero at least at one point in the interval $(-\infty,\infty)$.

The Wronskian of the fundamental system of solutions of the differential equation (a) or (b) is

$$W=\frac{c}{p(x)},$$

where $c\neq 0$ is a convenient constant depending on the choice of the solutions (Greguš 1965; Abdel Karim, 1973).

The relations between the solutions of the differential equations (a) and (b) can be interpreted as follows:

Let y_1, y_2, y_3, y_4 be arbitrary linearly independent solutions of the differential equation (a). Then the function

$$z(x)=w[y_1,y_2,y_3,y_4](x)=p(x)\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ (py_1'')' & (py_2'')' & (py_3'')' & (py_4'')' \end{vmatrix}$$

is a solution of the differential equation (b).

Furthermore, if z_1, z_2, z_3, z_4 are arbitrary linearly independent solutions of the differential equation (b), then the function

$$y(x)=w^*[z_1,z_2,z_3,z_4](x)=p(x)\begin{vmatrix} z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \\ z_1'' & z_2'' & z_3'' & z_4'' \\ z_1''' & z_2''' & z_3''' & z_4''' \end{vmatrix}$$

is a solution of the differential equation (a).

Theorem 1 (*Existence theorem*)

Let $y(x)$ be an arbitrary solution of the differential equation (a). Then there exist four solutions $z_1(x)$, $z_2(x)$, $z_3(x)$, $z_4(x)$ of the differential equation (b) such that the relation $y(x)=w^*[z_1,z_2,z_3,z_4](x)$ holds.

On the other hand, if $z(x)$ is an arbitrary solution of the differential equation (b), then there exist four solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ of the differential equation (a) such that the relation $z(x)=w[y_1,y_2,y_3,y_4](x)$ holds.

For the proof see (the author, 1972).

2. Basic Properties of The Solutions

Now we shall derive some basic properties of the solutions of the differential equations (a) and (b).

Theorem 2.

All the solutions $y(x)$ of the differential equation (a) with the property

i) $y(a)=y'(a)=y''(a)=(py'')'(a)=0$, $(py'')''(a)\neq 0$, $-\infty<a<\infty$ are linearly dependent.

Proof.

Let $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$, $y_5(x)$ be a fundamental system of solutions of the differential equation (a). Evidently

$$\bar{y}(x)=\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) & y_5(x) \\ y_1(a) & y_2(a) & y_3(a) & y_4(a) & y_5(a) \\ y_1'(a) & y_2'(a) & y_3'(a) & y_4'(a) & y_5'(a) \\ y_1''(a) & y_2''(a) & y_3''(a) & y_4''(a) & y_5''(a) \\ (py_1'')'(a) & (py_2'')'(a) & (py_3'')'(a) & (py_4'')'(a) & (py_5'')'(a) \end{vmatrix}$$

is a solution of the differential equation (a) with the property

$$\bar{y}(a)=\bar{y}'(a)=\bar{y}''(a)=(p\bar{y}'')'(a)=0,\ (p\bar{y}'')''(a)\neq 0.$$

Suppose now that y(x) is an arbitrary solution of the differential equation (a) with the property

$y(a)=y'(a)=y''(a)=(py'')'(a)=0,\ (py'')''(a)=k\neq 0$. Then y(x) can be written as

$$y(x)=\sum_{r=1}^{5} c_r y_r(x),$$

where the constants c_r satisfy the system of equations

$$\sum_1^5 c_r y_r(a)=0\ ,\ \sum_1^5 c_r y_r'(a)=0,\ \ \sum_1^5 c_r y_r''(a)=0$$

$$\sum_1^5 c_r (py_r'')'(a)=0,\ \ \sum_1^5 c_r (py_r'')''(a)=k\neq 0.$$

Evaluating the constants c_r, we find

$$y(x)=\frac{k}{(p\bar{y}'')''(a)}\bar{y}(x)\ .$$

Hence, the proof is complete.

Similarly, it can be shown, that all the solutions of the differential equation (a) with the alternative properties

ii) $y(a)=y'(a)=y''(a)=(py'')''(a)=0,\ \ (py'')'(a)\neq 0$

or

iii) $y(a)=y'(a)=(py'')'(a)=(py'')''(a)=0,\ \ y''(a)\neq 0$

or

iv) $y(a)=y''(a)=(py'')'(a)=(py'')''(a)=0,\ \ y'(a)\neq 0$

or

v) $y'(a)=y''(a)=(py'')'(a)=(py'')''(a)=0,\ \ y(a)\neq 0,$

$-\infty < a < \infty$ are linearly dependent. (Greguš, 1963; Abdel Karim, 1973).

Likewise the proof of theorem 2, it can be proved

Theorem 3

All the solutions z(x) of the differential equation (b) with the alternative properties

i') $$z(a)=z'(a)=z''(a)=z'''(a)=0, \quad (pz''')'(a)\neq 0$$

or

ii') $$z(a)=z'(a)=z''(a)=(pz''')'(a)=0, \quad z'''(a)\neq 0$$

or

iii') $$z(a)=z'(a)=z'''(a)=(pz''')'(a)=0, \quad z''(a)\neq 0$$

or

iv') $$z(a)=z''(a)=z'''(a)=(pz''')'(a)=0, \quad z'(a)\neq 0$$

or

v') $$z'(a)=z''(a)=z'''(a)=(pz''')'(a)=0, \quad z(a)\neq 0,$$

$-\infty < a < \infty$ are linearly dependent.

Let y_1, y_2, y_3, y_4, y_5 be a fundamental system of solutions of the differential equation (a), which satisfy at the point $a\in(-\infty,\infty)$ the following initial conditions:

$$y_1(a)=y_1'(a)=y_1''(a)=(py_1'')'(a)=0, \quad (py_1'')''(a)\neq 0,$$

$$y_2(a)=y_2'(a)=y_2''(a)=(py_2'')''(a)=0, \quad (py_2'')'(a)\neq 0,$$

$$y_3(a)=y_3'(a)=(py_3'')'(a)=(py_3'')''(a)=0, \quad y_3''(a)\neq 0,$$

$$y_4(a)=y_4''(a)=(py_4'')'(a)=(py_4'')''(a)=0, \quad y_4'(a)\neq 0,$$

$$y_5'(a)=y_5''(a)=(py_5'')'(a)=(py_5'')''(a)=0, \quad y_5(a)\neq 0.$$

Then there holds

Theorem 4

Every solution y(x) of the differential equation (a) with the alternative properties

$y(a)=0$, or $y'(a)=0$, or $y''(a)=0$, or $(py'')'(a)=0$ or $(py'')''(a)=0$; $a\in(-\infty,\infty)$

can be written in the form

$$y=\sum_{1}^{4}c_r y_r,\quad \text{or } y=\sum_{\substack{1\\(r\neq 4)}}^{5}c_r y_r,\quad \text{or } y=\sum_{\substack{1\\(r\neq 3)}}^{5}c_r y_r,$$

or
$$y=\sum_{\substack{1\\(r\neq 2)}}^{5}c_r y_r,\quad \text{or } y=\sum_{2}^{5}c_r y_r$$

respectively.

Analogous statement holds for the differential equation (b).

3. Existence of Solutions Without Zeros

In this paragraph and in the next one it will be assumed that $q(x)\geqq 0$ for $x\in(-\infty,\infty)$ and that $q(x)\equiv 0$ does not hold in any interval.

For the solutions of the differential equation (a) the following integral identities hold:

(1)
$$y(py'')''-\int_a^x[y'(py'')''-qy^2]dt=\text{const.}$$

(2)
$$(py'')''+\int_a^x qy\,dt=\text{ const.}$$

Similarly, the integral identities for the differential equation (b) have the form

(1')
$$z(pz''')'-\int_a^x[z'(pz''')'+qz^2]dt=\text{ const.}$$

(2')
$$(pz''')'-\int_a^x qz\,dt=\text{const. },$$

$$a\in(-\infty,\infty),\ x\in(-\infty,\infty)$$

The following theorems are established:

Theorem 5

Let y(x) be the solution of the differential equation (a) satisfying at the point $a\in(-\infty,\infty)$ the alternative initial conditions i) – v). Then y(x), y′(x), y″(x), (py″)′(x), (py″)″ (x) have no zero point to the left side of a.

Proof. Case i).

Let y(x) be the solution of the differential equation (a) satisfying the initial conditions i). Let $(py'')''(a)>0$ and suppose on the contrary, that e.g. $(py'')''(x_1)=0$, where $x_1<a$ is the first zero point of $(py'')''$ to the left of a. Since $(py'')'$ starts at a with positive slope, then it must attain its first minimum to the left of a before it would have its first zero point to the left of a at x_2 (say). Furthermore, since py″ has a double zero point at a and its slope in the left neighbourhood of a is negative, then it has to reach its first maximum to be left of a before it would have its first zero point to the left of a at x_3 (say). Also y″[y′] will have its first zero point to the left of a before y′[y] has this property at the point x_4 $[x_5]$ (say). It follows that

(3) $$\operatorname{sgn} y=-\operatorname{sgn} y'=\operatorname{sgn} y''=-\operatorname{sgn}(py'')'=\operatorname{sgn}(py'')'' \text{ in } (x_1,a),$$

where $-\infty<x_5<x_4<x_3<x_2<x_1<a<\infty$.

Setting $x=x_1$ in the integral identity (1), we get

$$0>[\,y(py'')''\,]_a^{x_1}-\int_a^{x_1}[\,y'(py'')''-qy^2\,]\,dt=0,$$

which is a contradiction. Then $(py'')''$ has no zero point to the left side of a. From the properties of the monotonic functions it follows that (py″)′, y″, y′, y have no zero point to the left side of a.

Case ii).

Suppose that $(py'')'(x_1)=0$, where $x_1<a$ is the first zero point of $(py'')'$ to the left of a. Then

(4) $$\operatorname{sgn} y=-\operatorname{sgn} y'=\operatorname{sgn} y''=-\operatorname{sgn}(py'')' \text{ in } (x_1,a).$$

Integrating (2) from a to x_1, we obtain the contradiction

$$0\neq-(py'')'(a)+\int_a^{x_1}(x_1-t)qy\,dt=0,$$

since y keeps its sign in (x_1,a). Then $(py'')'$ and hence y'', y', y have no zero point to the left of a. By virtue of (2), it can be also shown that $(py'')''$ has no zero point for $x<a$.

Case iii).

Supposing that $x_1<a$ is the first zero point of y'' to the left of a, then

$$\operatorname{sgn} y=-\operatorname{sgn} y'=\operatorname{sgn} y'' \quad \text{in } (x_1,a).$$

Double integration of (2) from a to x gives

(5) $$y''+\frac{1}{2p}\int_a^x (x-t)^2 q(t)\, y(t)\, dt=$$

$$(py'')''(a)\frac{(x-a)^2}{2p}+(py'')'(a)\frac{x-a}{p}+(py'')(a)\frac{1}{p}.$$

Setting $x=x_1$ in (5), we get

$$0\neq-(py'')(a)+\frac{1}{2}\int_a^{x_1}(x_1-t)^2 q(t)y(t)\, dt=0.$$

It follows that y'' and also y', y have no zero point for $x<a$. Using (2) and its integration from a to x, we find that $(py'')''$ and $(py'')'$ respectively have no zero point for $x<a$.

Case iv).

Let $x_1<a$ be the first zero point of y' to the left of a. Then $\operatorname{sgn} y=-\operatorname{sgn} y'$ in (x_1,a). Integration of (5) from a to x_1 shows that y' and consequently y have no zero point for $x<a$. Furthermore, there holds $\operatorname{sgn} y''=-\operatorname{sgn}(py'')'=\operatorname{sgn}(py'')''$ in $(\bar{x},a)$, where $\bar{x}<a$ is assumed to be the first zero point of $(py'')''$ to the left of a. Setting $x=\bar{x}$ in(2), it follows that $(py'')''$ and also $(py'')'$, y'' have no zero point for $x<a$.

The last case can be similarly proved.

Theorem 6

Let z(x) be the solution of the differential equation (b) with the alternative initial values i')$-$v') at the point $a\in(-\infty,\infty)$. Then $z(x)$, $z'(x)$, $z''(x)$, $z'''(x)$, $(pz''')'(x)$ have no zero point to the right side of a.

Proof

Let z(x) be the solution of the differential equation (b) satisfying the initial conditions i'). Suppose on the contrary that e.g. $(pz''')'(x_1)=0$, where $x_1>a$ is the first zero point of $(pz''')'$ to the right of a. Likewise the proof of case i), it can be shown that $\operatorname{sgn} z=\operatorname{sgn} z'=\operatorname{sgn} z''=\operatorname{sgn} z'''=\operatorname{sgn}(pz''')'$ in (a,x_1). Setting $x=x_1$ in the integral identity

(1′), we find that $(pz''')'$ and hence z''', z'', z', z have no zero to the right side of a. The other cases can be similarly proved.

We note here, that other similar types of theorems can be also given. (Abdel Karim 1977).

4. The Behaviour of The Solutions

By means of the results of the preceding paragraph, we shall investigate the behaviour of the solutions of the differential equations (a) and (b).

There holds

Theorem 7

Let $0 < p(x) \leq m$ for $x \in (-\infty, \infty)$, where m is a constant. Let y(x) be the solution of the differential equation (a) satisfying at the point $a \in (-\infty, \infty)$ the alternative initial conditions i) or iii) or v), in which the sign $\neq$ is replaced by $>$. Then there hold

$$\lim_{x \to -\infty} y(x) = \lim_{x \to -\infty} y''(x) = +\infty,$$

$$\lim_{x \to -\infty} y'(x) = \lim_{x \to -\infty} (py'')'(x) = -\infty.$$

There exists also $\lim_{x \to -\infty} (py'')''(x)$, which is finite or $+\infty$.

Proof

From theorem 5, it follows that $y, y', y'', (py'')', (py'')''$ have no zero point for $x < a$, and there hold

$$y > 0, \; y' < 0, \; y'' > 0, \; (py'')' < 0, \; (py'')''(a) > 0 \quad \text{for } x < a.$$

Case i).

Let y(x) be the solution of the differential equation (a) with the initial conditions i). Successive integration of the integral identity (2) from a to x leads to the following inequalities, which are valid for $x < a$

$$(py'')'(x) < (py'')''(a)\,(x-a), \quad y''(x) > \frac{(py'')''(a)}{2m}(x-a)^2$$

$$y'(x) < \frac{(py'')''(a)}{3!m}(x-a)^3, \quad y(x) > \frac{(py'')''(a)}{4!m}(x-a)^4.$$

It follows from these inequalities that $y \to +\infty, y' \to -\infty, y'' \to +\infty, (py'')' \to -\infty$ as $x \to -\infty$.

Referring to the differential equation (a) it is evident that $(py'')'' \leqq 0$ for $x < a$, where the equality sign holds only for the isolated points. Therefore $(py'')''$ is a positive monotonic non-increasing function in $(-\infty, a)$ and there exists $\lim_{x \to -\infty} (py'')''$.

Case iii).

Let y(x) be the solution of the differential equation (a) with the initial conditions iii). Then the integral identity (2) gives for $x < a$

$$(py'')'(x) = \int_x^a (x-t)\, q(t)\, y(t)\, dt, \quad y''(x) > \frac{1}{2m} \int_x^a (x-t)^2 q(t)\, y(t)\, dt,$$

$$y'(x) < \frac{(py'')(a)}{m}(x-a), \quad y(x) > \frac{(py'')(a)}{2m}(x-a)^2,$$

from which the requirement follows.

The remaining case can be similarly proved.

Theorem 8

Let $0 < p(x) \leqq m$ for $x \in (-\infty, \infty)$, where m is a constant. Let y(x) be the solution of the differential equation (a) satisfying at the pomt $a \in (-\infty, \infty)$ the alternative initial conditions ii) or iv), in which the sign $\neq$ is replaced by $>$. Then

$$\lim_{x \to -\infty} y(x) = \lim_{x \to -\infty} y''(x) = -\infty,$$

$$\lim_{x \to -\infty} y'(x) = \lim_{x \to -\infty} (py'')'(x) = +\infty$$

and there exists also $\lim_{x \to -\infty} (py'')''(x)$ which is finite or $-\infty$.

Proof

Application of theorem 5 shows that

$$y < 0,\ y' > 0,\ y'' < 0,\ (py'')' > 0,\ (py'')'' < 0 \text{ for } x < a.$$

Let y(x) be the solution of the differential equation (a) with the alternative initial conditions ii) or iv). Then the integral identity (2) leads to the following inequalities, which are valid for $x < a$

$$(py'')'(x) > \int_x^a (x-t)q(t)y(t)\,dt, \quad y''(x) < \frac{(py'')'(a)}{m}(x-a),$$

$$y'(x) > \frac{(py'')'(a)}{2m}(x-a)^2, \quad y(x) < \frac{(py'')'(a)}{3!m}(x-a)^3,$$

or

$$(py'')'(x) = \int_x^a (x-t)\,q(t)\,y(t)\,dt\,, \qquad y''(x) \leqq \frac{1}{2m}\int_x^a (x-t)^2\,q(t)\,y(t)\,dt\,,$$

$$y'(x) > \frac{1}{3!m}\int_x^a (x-t)^3\,q(t)\,y(t)\,dt\,, \qquad y(x) < y'(a)\,(x-a),$$

respectively. This completes the proof.

By the same procedure used in the theorems 7 and 8, it is possible to prove.

Theorem 9

Let $0 < p(x) \leqq m$ for $x \in (-\infty, \infty)$. Let $z(x)$ be the solution of the differential equation (b) with the alternative initial conditions i')$-$v') at the point $a \in (-\infty, \infty)$, in which the sign $\neq$ is replaced by $>$. Then there hold

$$\lim_{x\to\infty} z(x) = \lim_{x\to\infty} z'(x) = \lim_{x\to\infty} z''(x) = \lim_{x\to\infty} z'''(x) = +\infty.$$

There exists also $\lim\limits_{x\to\infty} (pz''')'(x)$ which is finite or $+\infty$.

5. The Comparison Theorem

With the differential equation (a) we consider also the differential equation

$$(a_1) \qquad (p(x)u'')''' + q_1(x)u = 0,$$

where $p(x) > 0$ and $q_1(x)$ are continuous functions of $x \in (-\infty, \infty)$.

We need the following

Lemma.

Every solution $y(x)$ of the differential equation (a) can be written in the form

$$(6) \qquad y(x) = u(x) + \int_a^x (q_1(t) - q(t))\,p(t)\,W(x,t)\,y(t)\,dt\,,$$

where u(x) is a solution of the differential equation (a_1) with the same initial conditions at the point $a \in (-\infty, \infty)$ as y(x), W(x,t) is a function of the form

$$
(7)\quad W(x,t) = \begin{vmatrix} u_1(x) & u_2(x) & u_3(x) & u_4(x) & u_5(x) \\ u_1(t) & u_2(t) & u_3(t) & u_4(t) & u_5(t) \\ u_1'(t) & u_2'(t) & u_3'(t) & u_4'(t) & u_5'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) & u_4''(t) & u_5''(t) \\ (pu_1'')'(t) & (pu_2'')'(t) & (pu_3'')'(t) & (pu_4'')'(t) & (pu_5'')'(t) \end{vmatrix}
$$

and $u_r(x)$ $(r = 1,2,3,4,5)$ from a fundamental system of solutions of the differential equation (a_1) whose Wronskian is equal to $\dfrac{1}{p(x)}$.

Proof

By means of the method of variation of the parameters applied on the differential equation

$$(py'')''' + q_1 y \;=\; (q_1 - q)y \;,$$

we write the solution of the differential equation (a) in the form

$$y \;=\; \sum_1^5 c_r(x) u_r(x) \;,$$

where the functions $c_r(x)$ are obtained from the system of equations

$$\sum_1^5 c_r' u_r = 0, \quad \sum_1^5 c_r' u_r' = 0, \quad \sum_1^5 c_r' u_r'' = 0,$$

$$\sum_1^5 c_r' (pu_r'')' = 0, \quad \sum_1^5 c_r' (pu_r'')'' = (q_1 - q)y.$$

Hence, the proof is complete. (Abdel Karim, 1972; Greguš and Abdel Karim, 1970).

Theorem 10 (*Comparison theorem*).

Let $0 \leqq q_1(x) < q(x), -\infty < x < \infty$. Let y(x) and u(x) be two solutions of the differential equations (a) and (a_1) with the initial values at the point $a \in (-\infty, \infty)$:

$$(8)\qquad \begin{aligned} &y(a)=y'(a)=y''(a)=(py'')'(a)=0, \ (py'')''(a)\neq 0. \\ &u(a)=u'(a)=u''(a)=(pu'')'(a)=0, \ (pu'')''(a)\neq 0. \end{aligned}$$

If $x_1>a$ is the first zero point of u(x) on the right of a, then y(x) has at least one zero point on (a,x_1).
On the other hand, if y(x) has no zero point on (a,∞), then u(x) has also no zero point on (a,∞).

Proof

Let y(x) and u(x) be the solutions of the differential equations (a) and (a_1) with the initial values (8) and let without loss of generality $(py'')''(a)=(pu'')''(a)>0$ (see theorem 2). Referring to the preceding lemma, the relation (6) holds between the two solutions y(x) and u(x). Evidently the function W(x,t), which is defined in (7), is for fixed t a solution of the differential equation (a_1) with the properties

$$W(t,t)=W_x'(t,t)=W_x''(t,t)=[pW_x'']_x'(t,t)=0,$$

$$[pW_x'']_x''(t,t)=\frac{1}{p(t)}>0,$$

and therefore $W(x,t)\geqq 0$ for $a\leqq t\leqq x\leqq x_1$.

Let $x_1>a[x_2>a]$ be the first zero point on the right of a of the solution u(x)[y(x)]. Then it follows from the relation (6), that $a<x_2<x_1$.

The second part follows also by using (6).

Furthermore, there holds

Theorem 11

Let $0\leqq q_1(x)<q(x), -\infty<x<\infty$. Let y(x) and u(x) be two solutions of the differential equations (a) and (a_1) satisfying at the point $a\in(-\infty,\infty)$ the alternative initial conditions

$$y(a)=y'(a)=y''(a)=(py'')''(a)=0, \quad (py'')'(a)\neq 0$$

$$u(a)=u'(a)=u''(a)=(pu'')''(a)=0, \quad (pu'')'(a)\neq 0$$

or

$$y(a)=y'(a)=(py'')'(a)=(py'')''(a)=0, \quad y''(a)\neq 0$$

$$u(a)=u'(a)=(pu'')'(a)=(pu'')''(a)=0, \quad u''(a)\neq 0$$

or

$$y(a)=y''(a)=(py'')'(a)=(py'')''(a)=0, \quad y'(a)\neq 0$$

$$u(a)=u''(a)=(pu'')'(a)=(pu'')''(a)=0, \quad u'(a)\neq 0$$

or

$$y'(a)=y''(a)=(py'')'(a)=(py'')''(a)=0, \quad y(a)\neq 0$$

$$u'(a)=u''(a)=(pu'')'(a)=(pu'')''(a)=0, \quad u(a)\neq 0.$$

If $x_1 > a$ is the first zero point of u(x) on the right of a, then y(x) has at least one zero point on (a,x_1).

On the other hand, if y(x) has no zero point on (a,∞), then so does u(x) on (a,∞).

The analogues comparison theorems between two differential equations of the form (b) can be similarly proved.

6. Concerning The Zeros of The Solutions

Let us consider the differential equation

$$(\bar{a}) \qquad (p(x)y'')''' + q(x,\lambda)y = 0,$$

where $p(x)>0$ is a continuous function $x\in(-\infty,\infty)$ and $q(x,\lambda)\geqq 0$ is a continuous function of $x\in(-\infty,\infty)$ and $\lambda\in(\Lambda_1,\Lambda_2)$, and $q\equiv 0$ does not hold in any interval.

Then there holds

Theorem 12

Let $\lim_{\lambda\to\Lambda_2} q(x,\lambda)=+\infty$ holds uniformly for all $x\in(-\infty,\infty)$. Let $a<b\in(-\infty,\infty)$ be given numbers. Further let $y(x,\lambda)$ be the solution of the differential equation $(\bar{a})$ satisfying at the point $a\in(-\infty,\infty)$ the alternative initial conditions i)-v). Then there exists a parameter $\lambda\in(\Lambda_1,\Lambda_2)$ such that $y(x,\lambda)$ has a farther zero point in (a,b).

Proof

Let y(x) be the solution of the differential equation (ā) satisfying e.g. the initial conditions ii), and let without loss of generality $(py'')'(a)=1$. We compare the differential equation (ā) with the equation

$$(p(x)u'')'''=0\ , \tag{9}$$

which has a fundamental system of solutions

$$u_1=\tfrac{1}{2}\int_a^x\frac{(x-t)(t-a)^2}{p(t)}\,dt,\quad u_2=\int_a^x\frac{(x-t)(t-a)}{p(t)}\,dt, \tag{10}$$

$$u_3=\int_a^x\frac{x-t}{p(t)}\,dt,\quad u_4=x-a,\quad u_5=1.$$

whose Wronski determinant is $W(x)=\dfrac{1}{p(x)}$. Then $u_2(x)$ satisfies at the point a the same initial conditions as y(x). From the preceding lemma, it follows that y can be written in the form

$$y(x,\lambda)=\int_a^x\frac{(x-t)(t-a)}{p(t)}\,dt-\int_a^x q(t,\lambda)\,p(t)\ .\ (W(x,t)\ y(t,\lambda)\,dt, \tag{11}$$

where W(x,t) is defined in (7), and $u_r(x)(r=1,2,3,4,5)$ form a fundamental system of solutions of the differential equation (9). For fixed t the function $W(x,t)=\bar{u}(x)$ is a solution of the differential equation (9) with the properties

$$\bar{u}(t)=\bar{u}'(t)=\bar{u}''(t)=(p\bar{u}'')'(t)=0,\quad (p\bar{u}'')''(t)=W(t)=\frac{1}{p(t)}>0\ \cdot$$

Therefore

$$W(x,t)=\frac{1}{2p(t)}\int_t^x\frac{(x-s)(s-t)^2}{p(s)}\,ds\,.$$

Substituting in (11), we get

$$y(x,\lambda)=\int_a^x\frac{(x-t)(t-a)}{p(t)}dt-\tfrac{1}{2}\int_a^x q(t,\lambda)\quad y(t,\lambda)\left(\int_t^x\frac{(x-s)(s-t)^2}{p(s)}ds\right)dt.$$

Supposing on the contrary that $y(x,\lambda)$ has no zero point for $x\in(a,b)$ and $\lambda\in(\Lambda_1,\Lambda_2)$, then $y(x,\lambda)>0$ for $a<x<b$ and $\Lambda_1<\lambda<\Lambda_2$. But the function $y(b,\lambda)$ is continuous in $\lambda\in(\Lambda_1,\Lambda_2)$ and with increasing $\lambda\to\Lambda_2$ it will be negative, which leads to a contradiction.

Consider the solution y(x) of the differential equation (a) satisfying the initial conditions iii), and let $y''(a)=1$. The differential equation (9) has a fundamental system of solutions.

$$u_1=\frac{1}{2}\int_a^x\frac{(x-t)(t-a)^2}{p(t)}\,dt,\quad u_2=\int_a^x\frac{(x-t)(t-a)}{p(t)}\,dt,$$

$$u_3=p(a)\int_a^x\frac{x-t}{p(t)}\,dt,\quad u_4=x-a,\quad u_5=1,$$

whose Wronskian is $W(x)=\frac{p(a)}{p(x)}$. Since u_3 and y satisfy the same initial conditions at the point a, then y can be written as

$$y(x,\lambda)=p(a)\int_a^x\frac{x-t}{p(t)}\,dt-\frac{1}{p(a)}\int_a^x q(t,\lambda)\,p(t)\quad W(x,t)y(t,\lambda)\,dt, \tag{12}$$

where

$$W(x,t)=\frac{p(a)}{2p(t)}\int_t^x\frac{(x-s)(s-t)^2}{p(s)}\,ds.$$

Consequently

$$y(x,\lambda)=p(a)\int_a^x\frac{x-t}{p(t)}\,dt-\frac{1}{2}\int_a^x q(t,\lambda)\,y(t,\lambda).\quad\left(\int_t^x\frac{(x-s)(s-t)^2}{p(s)}ds\right)dt, \tag{12}$$

from which the requirement follows.

Suppose that y(x) is the solution of the differential equation ($\bar{a}$) with the initial conditions v) and let $y(a)=1$. We compare with the differential equation (9), whose fundamental system of solution is (10).

Evidently u_5 and y satisfy the same initial conditions at the point a. Analogous to (12) we obtain

$$y(x,\lambda)=1-\frac{1}{2}\int_a^x q(t,\lambda)\,y(t,\lambda).\quad\left(\int_t^x\frac{(x-s)(s-t)^2}{p(s)}ds\right)dt.$$

Hence, the proof is complete.

The other cases can be similarly proved.

References

Abdel Karim, Rahmi I.I. (1972) On the zeros of solutions of certain differential equations of the fifth order. *Acta Math.. Academie Sci.* **13**. 335-340.

——— (1973) The properties of the solutions of certain quasilinear differential equations of the fifth order. *Periodica Math.*, **4**. 91-102.

——— (1977) On the distribution of zeros of the solutions of certain quasilinear differential equations of the third order. *Bull. Fac. Sci. Riyadh Univ.*, **8**. 475-488.

Greguš, M. (1963) Über die lineare homogene Differentialgleichung dritter Ordnung. *Wiss. Z. Univ. Halle-Wittenberg Math. Natur.*, **12**. 265-286.

——— (1965) Über die Eigenschaften der Lösungen einiger quasilinear Gleichungen 3. Ordnung. *Acta F.R.N. Univ. Comen. Math.*, **10**. 11-22.

——— and **Rahmi I.I. Abdel Karim** (1970) Boundedness of the solutions of the differential equation $(py')'' + (py')' + ry = 0$, *Proc. Math. Phys. Soc. U.A.R.* (Egypt), **32**, 107–110.

عن حلول المعادلات التفاضلية الشبه خطية من الرتبة الخامسة

رحمي إبراهيم إبراهيم عبد الكريم

قسم الرياضيات ، كلية العلوم ، جامعة الرياض ، الرياض ، المملكة العربية السعودية .

درس المؤلف في هذا البحث الخواص الأساسية لحلول معادلات تفاضلية معينة شبه خطية من الرتبة الخامسة . ولقد أعطيت الشروط الكافية لكي تكون حلول هذه المعادلات بدون أصفار . وبعد ذلك اشتقت نظرية المقارنة ، كما درست مسألة تتعلق بتوزيع أصفار حلول هذه المعادلات التفاضلية .