تعديل دالة لاقرانج

صالح عبد الرحمن العذل*

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*أستاذ مساعد في قسم الهندسة الميكانيكية – كلية الهندسة – جامعة الرياض .

GENERATION OF A MODIFYING LAGRANGIAN

Saleh A. Athel*

The aim of this note is to generate a Lagrangian function which can be added to the Lagrangian of any dynamical system to get rid of certain terms or to modify its form. This modifying Lagrangian plays the same role with respect to the Lagrangian as the generating function associated with the Hamiltonian but it is easier to generate.

1. INTRODUCTION

The Lagrangian formulation of the equations of motion of a dynamical system are used quite often in the analysis of these systems. For conservative systems, the Hamiltonian formulation is used with the Hamiltonian derived from the Lagrangian. In some cases, the Lagrangian contains some terms which have no effect whatsoever on the equations of motion and can therefore be discarded from the outset. In other cases, it is preferable to modify the Lagrangian before deriving the Hamiltonian. The purpose of this note is to show how to spot these superfluous terms in the Lagrangian or to modify it by adding certain terms to it. This modifying Lagrangian is analagous to the generating function for canonical transformations but it simplifies the generation of this function in some cases while in other cases it may replace it and as we shall see it is easy to generate.

2. ANALYSIS

The equations of motion of any dynamical system are given by [1]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k \qquad k = 1, \ldots, n \qquad (1)$$

where q_k, $k=1,\ldots, n$ are the generalized coordinates, Q_k is the generalized non-potential active force associated with q_k and $L = L(q_1,\ldots, q_n, \dot{q}_1,\ldots \dot{q}_n, t)$ is the Lagrangian of the system.

The aim of this note is to answer the following question: is it possible to modify L, i.e., deleting some terms or adding new ones, apriori, i.e., before solving Eq. (1) for q_k and q_k, $k = 1, \ldots, n$ such that Eq. (1) will still be satisfied ? The answer to this question is yes and the purpose of this note is to show how to generate a modifying Lagrangian L^* to be added to L without affecting Eq. (1), i.e., such that

$$\frac{d}{dt}\frac{\partial (L + L^*)}{\partial \dot{q}_k} - \frac{\partial (L + L^*)}{\partial q_k} = Q_k \quad k = 1, \ldots, n \quad (2)$$

from which it follows that

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_k} - \frac{\partial L^*}{\partial q_k} = 0 \qquad k = 1, \ldots, n \qquad (3)$$

we want to generate a function L^* such that Eq. (3) is satisfied identically regardless of the values of q_k and $\dot{q}_k$, $k = 1, \ldots, n$.

Noting that $L^*(q_1, \ldots, q_n, q_1, \ldots, q_n, t)$, Eq (3) can be rewritten as follows:

$$\sum_{i=1}^{n} \frac{\partial^2 L^*}{\partial \dot{q}_k \partial \dot{q}_i}\ddot{q}_i + \sum_{i=1}^{n}\frac{\partial^2 L^*}{\partial \dot{q}_k \partial q_i}\dot{q}_i + \frac{\partial^2 L^*}{\partial \dot{q}_k \partial t} - \frac{\partial L^*}{\partial q_k} = 0 \qquad k = 1, \ldots, n \qquad (4)$$

Since L^* does not depend explicitly on $\ddot{q}_k$, $k = 1, \ldots, n$ and we want Eq. (4) to be satisfied identically, then we must have

* *Assistant Professor, Mechanical Engineering Department, College of Engineering, University of Riyad*

$$\frac{\partial^2 L^*}{\partial \dot{q}_k \, \partial \dot{q}_i} = 0 \qquad i,k = 1, \ldots, n \qquad (5)$$

The only possible solution to Eq. (5) is

$$L^* = \sum_{i=1}^{n} G_i (q_1, \ldots, q_n, t)\, \dot{q}_i + F (q_1, \ldots, q_n, t) \quad (6)$$

Substitution from Eq. (6) into Eq. (4) yields

$$\sum_{i=1}^{n} \frac{\partial G_k}{\partial q_i} q_1 + \frac{\partial G_k}{\partial t} - \sum_{i=1}^{n} \frac{\partial G_i}{\partial q_k} q_i - \frac{\partial F}{\partial q_k} = 0$$

$$k = 1, \ldots, n \qquad (7)$$

Eq. (7) can be satisfied identically if

$$\frac{\partial G_k}{\partial q_i} = \frac{\partial G_i}{\partial q_k} \qquad i,k = 1, \ldots, n \qquad (8)$$

$$\frac{\partial G_k}{\partial t} = \frac{\partial F}{\partial q_k} \qquad k = 1, \ldots, n \qquad (9)$$

Eqs. (8) and (9) can always be satisfied if there exists a function $\phi (q_1, \ldots, q_n, t)$ and G_k, $k = 1, \ldots, n$ and F are chosen such that

$$G_k = \frac{\partial \phi}{\partial q_k} \qquad k = 1, \ldots, n \qquad (10)$$

and

$$F = \frac{\partial \phi}{\partial t} \qquad (11)$$

3. SUMMARY

The function

$$L^* = \sum_{i=1}^{n} G_i (q_1, \ldots, q_n, t)\, \dot{q}_i + F (q_1, \ldots, q_n, t)$$

can be added or subtracted from any Lagrangian provided that

$$\frac{\partial G_k}{\partial q_i} = \frac{\partial G_i}{\partial q_k} \qquad i,k = 1, \ldots, n$$

and

$$\frac{\partial G_k}{\partial t} = \frac{\partial F}{\partial q_k} \qquad k = 1, \ldots, n$$

The function $L^* (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ can be generated as follows:

1. Choose any suitable function $\phi (q_1, \ldots, q_n, t)$
2. Form $G_k (q_1, \ldots, q_n, t)$, $k=1, \ldots, n$ and $F (q_1, \ldots, q_n, t)$ as

$$G_k(q_1, \ldots, q_n, t) = \frac{\partial}{\partial q_k} \phi (q_1, \ldots, q_n, t)$$

and

$$F (q_1, \ldots, q_n, t) = \frac{\partial}{\partial t} \phi (q_1, \ldots, q_n, t)$$

3. Form $L (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ as

$$L^* (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) =$$

$$\sum_{i=1}^{n} G_i (q_1, \ldots, q_n, t)\, q_1 + F (q_1, \ldots, q_n, t)$$

It is worth noting that

$$L^* = \sum_{i=1}^{n} G_i \dot{q}_i + F$$

$$= \sum_{i=1}^{n} \frac{\partial \phi}{\partial q_i} \dot{q}_i + \frac{\partial \phi}{\partial t}$$

$$= \frac{d\phi}{dt}$$

That is, the addition to the Lagrangian of any complete derivative with respect to time does not change the equations of motion.

REFERENCES

1) Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill, New York, N.Y. 1970, p. 76