الصفائح المرتكزة على الأساسات المرنة

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يقدم البحث دراسات عن الصفائح المرتكزة على الأساسات المرنة ، لقد بني التحليل على أن الطبقة السفلية للتربة تمثل مجموعة من النوابض اللولبية مرتبطة بغشاء أفقي ذى قوة شد ثابتة ، لقد اقتصر البحث على مسائل التناظر المحوري والصفائح المستطيلة فقط .

تم الحصول على نتائج عددية لدراسة أثر العوامل الرئيسية الداخلة في هذه المسألة والتي تشمل :

١ – دليل مرونة شد الغشاء .

٢ – معامل المكافىء .

لقد وجد أن دليل المرونة يزداد بنقصان التواءآت العزوم في الصفيحة ، كذلك وجد أنه كلما ازداد معامل الشد صغرت الالتواءات داخل الصفيحة وبهذا يتم توزيع الحمل خارج منطقة الصفيحة ، يولد ازدياد صغير في معامل الشد عزوماً في الصفائح المحملة بانتظام ، غير أنه ينقص العزوم المتولدة من الحمل المركز على الصفيحة المرتكزة على الأساسات المرنة .

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PLATES ON ELASTIC FOUNDATIONS

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Studies of plates on elastic foundations are presented. The analysis is based on the representation of the subgrade as a system of vertical springs connected by a horizontal membrane with constant tension force. Axially symmetric problems as well as rectangular plates are considered.

Numerical results were obtained to study the effect of the main parameters that involved into the problem. These parameters are:

1. *The flexibility index.*
2. *The equivalent membrane tension modulus of the subgrade.*

It was found that as the flexibility index increases the deflections and moments in the plate decrease. Also the larger the tension modulus the smaller the deflections inside the plate are, thus distributing the load to the region outside the plate. A small increase in tension modulus creates moments in a uniformly loaded plate, but reduces the moments arising from a concentrated load on a plate on an elastic foundation.

Nomenclature

$A_o, B_o, C_o \ldots, Z_o$ = Stiffness factors, defined by Eq. 12

a = half the length of the plate.

b = half the width of the plate.

c = aspect ratio $= b/a$

D = plate stiffness factor $= Eh^3/12(1-\mu^2)$

E, μ = plate elasticity constants.

H_o = symbol of Hankel function

h = plate thickness

I_o and K_o = modified Bessel function of the second kind.

J_o = Bessel functions of the first kind.

k = spring's constant.

K = flexibility index $= Kb^4/D$

L_o = characteristic length $= (D/K)^{1/4}$

M = bending or twisting moments.

N = generalized shearing force acting on the plate and the subgrade

P = load intensity function in pounds per square inch

S = total shearing force acting on the subgrade

T = equivalent membrane tension modulus $= t/Kb^2$

t = constant membrane tension force.

u = strain energy

w = displacement in z-direction

v = total potential energy

α^2 = $K/2t$

ζ = X/b

η = y/b

∇^2 = Laplace operator

λ = $e^{i\Phi}/L_o$

$\bar{\lambda}$ = $e^{-i\Phi}/L_o$

λ_V, λ_H = distance between node points of a network

$\cos 2\Phi = -t/2\sqrt{kD}$

Ω = potential energy of external loads.

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1. Introduction

Analysis of plates on elastic foundation began in the last century when Winkler [1] proposed the simplest model of elastic foundation. The subgrade was replaced by vertical springs or, equivalently, by a heavy liquid. Practical applications of this theory to pavements design were made by Westergaard [2, 3, 4, 5]. Many particular solutions for axially symmetric problems were given by Schleicher [6]. Various solutions were given by Timoshenko [7] and others [8, 9].

The basis of another approach to the problem, considering the foundation as an elastic solid, was introduced by Boussinesq [10], who solved the problem of a semi-infinite homogeneous medium subjected to a concentrated normal load at the surface. Terazawa [11] presented the same solution using different approach. Newmark [12, 13] worked out influence charts computing stresses and vertical displacements in a homogeneous medium for any distribution of normal surface loading.

In the present century the problem has attracted the attention of a number of elasticians. Marguerre [14, 15], Biot [16], Pickett [17], Passer [18] and Serebrganyi [19] developed formulas for one layer of finite thickness. Hogg [20] and Holl [21] were first to analyze the problem of thin slabs of infinite size supported by a semi-infinite elastic solid and subjected to a symmetrical loading. The unsymmetrical loading was not worked out until 1947 by Volterra [22]. In 1943 and 1945 Burmister [23, 24] established the equations of stresses and displacements for the two and three-layer systems subjected to radially symmetric loading.

Extensive numerical data for vertical displacements and stresses for two and three-layer systems were presented by Burmister [25], Fox [26], Acum and Fox [27], Mehta and Veletsos [28] and Lemcoe [29].

The three dimensional elasticity problem of plates on elastic layer was studied by Vlassov and Leont'ev [30], using a variational method.

2. Methods of analysis

The differential equation of a medium thick plate supported by a system of vertical, parallel, tension-compression springs, connected by a horizontal membrance of constant force [30] is:

$$D\nabla^4 w_1 - t\,\nabla^2 w_1 + kw_1 = p \qquad (1)$$

where $(D\,\nabla^4 w_1)$ represent the action of the plate with flexural stiffness D and $(-\,t\nabla^2 w_1 + kw_1)$ characterize the behavior of the subgrade with constant membrane force t and spring's constant k.

The region outside the plate is characterized mathematically by (30):

$$\nabla^2 w_2 - \alpha^2 w_2 = 0 \qquad (2)$$

where $\alpha^2 = \frac{k}{2t}$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Both regions are connected through the following boundary conditions at the edges.

$$M_n = 0\,,\ w_1(a) = w_2(a) \quad \text{or}\ w_1(b) = w_2(b)$$
$$N_x(a) = S_x(a) \quad \text{or} \quad N_y(b) = S_y(b) \qquad (3)$$

where

M_n = the moment normal to the edge of the plate.

$w_1(a)$ and $w_1(b)$ = the deflection of the plate at the edge.

$w_2(a)$ and $w_2(b)$ = the deflection of the external region at the boundary between the two regions.

$N_x(a)$ and $N_y(b)$ = the generalized shearing force of the inner region at the boundary.

$S_x(a)$ and $S_y(b)$ = the generalized shearing force of the outer region at the boundary.

3. Axially Symmetric Problems:

For axially symmetric problems the deflections $w_1 = w_1(\mu)$ do not depend on the polar angle θ and the governing differential equations reduce to ordinary differential equations of the form:

$$D\,\nabla^4 w_1(\mu) - t\nabla^2 r\, w_1(\mu) + k\, w_1(\mu) = P \qquad (4a)$$

$$\nabla^2 r\, w_2(\mu) - \alpha^2 w_2(\mu) = 0 \qquad (4b)$$

where Laplace operator is

$$\nabla^2 r = \frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

The general solution of Eq. (4a) is:

$$w_1 = w_h + w_p$$

where w_p is the particular solution and w_h is the general solution of the homogenous equation:

$$\nabla^4_\mu\, w_1 - \frac{t}{D}\nabla^2_\mu\, w_1 + \frac{k}{D} w_1 = 0 \qquad (5)$$

Eq. (5) may be written in the form:

$$(\nabla^2 + \lambda^2)(\nabla^2 + \lambda^{-2})\, w_1 = 0 \qquad (6)$$

where $\lambda^2 . \lambda^{-2} = \frac{k}{D}$, $\lambda^2 + \lambda^{-2} = \frac{-t}{D}$

The general solution of Eq. (6) is the sum of the general solution of the two following equations :

$$(\nabla^2 + \lambda^2)\ w_a = 0$$
$$(\nabla^2 + \lambda^{-2})\ w_b = 0 \qquad (7)$$

Accordingly, the general solution of Eq. (4a) is:

$$w_1 = B_1 J_O(\lambda r) + B_2 H_O^{(1)}(\lambda r) + B_3 J_O(\bar{\lambda} r) + B_4 H_O^{(2)}(\bar{\lambda} r) + w_p \qquad (8)$$

and the general solution of Eq. (4b) is:

$$w_2 = B_5 I_O(\alpha r) + B_6 K_O(\alpha r) \qquad (9)$$

Eqs. (8) and (9), with the boundary conditions give the general solution of the axially symmetric problem of plates on elastic subgrades.

4. Rectangular Plates:

For rectangular plates it is more convenient to consider the governing differential equations in dimensionless coordinates. Introducing the following dimensionless quantities,

$$\eta = \frac{y}{b} \quad \zeta = \frac{x}{b} \quad \text{and} \quad c = \frac{b}{a}$$

where

2b = width of the plate in the y-direction

2a = length of the plate in the x-direction

c = aspect ratio

one obtains

$$\nabla^4 w_1 - 2T K \nabla^2 w_1 + K w_1 = \frac{pb^4}{D}$$

$$\nabla^2 w_2 - \frac{1}{2T} w_2 = 0 \qquad (10)$$

where

$$\nabla^2 = \frac{1}{b^2}\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right]$$

$K = \frac{kb^4}{D}$ = flexibility index

$T = \frac{t}{kb^2}$ = equivalent membrane tension modulus of the subgrade.

The solution of Eq. (10) is obtained by use of the finite difference method. The differential equations are replaced by their finite difference approximations rather than satisfying the differential equation and the boundary conditions at every point the difference aquation is satisfied at the node points of superposed grid.

Finite difference operators are normally derived by direct substitution of appropriate difference expressions into the governing differential equations, however it is difficult to use this procedure for certain unusual boundary conditions. The difficulty can be circumvented by the application of the energy method to a physical model of the problem.

5. Description of the Model:

The model consists of a plate analog and an elastic layer represented by a system of vertical parallel, tension-compression springs which are connected by a horizontal membrane. The plate analog is composed of rigid bars connecting elastic hinges with torsion springs attached to adjacent parallel bars. This analog has the following properties:

1. The external loads are concentrated at the elastic hinges
2. The resultants of direct stresses and vertical shearing stresses are bending moments and shearing forces are acting at the elastic hinge and at the end of each bar.
3. The resultant of the horizontal shearing stresses are twisting moments concentrated in the torsion springs.

6. The Derivation of the Difference Operators:

The theorem of stationary potential energy applied to the deflected configuration of the model yields an equation of the form

$$\delta V = \delta\,(U + \Omega) = 0 \qquad (11)$$

where

V = total potential energy of the model

U = strain energy of the model

Ω = potential energy of external loads

The quantities V, U and Ω are expressed in terms of the generalized coordinates W_i, i.e. the deflections at the node points of the difference net.

The difference operator at a point is obtained by minimization of the total potential energy with respect to the generalized coordinate at that point.

The difference operators for rectangular plates resting freely on the subgrade are given below.

$$D_o = \left[-4(1+c^2) - \frac{2t\lambda^2 H}{D} c^2 \right]$$

$$E_o = D_o c^2$$

$$F_o = \left[6 + 8c^2 + 6c^4 + \frac{4t\lambda^2 H}{D} c^2 (1+c^2) + \frac{k\lambda^4 v}{D} \right]$$

$$G_o = \mu c^2$$

$$H_o = \left[-2c^4 - 4c^2 + 2\mu c^2 - \frac{2t\lambda^2 H}{D} c^4 \right]$$

$$I_o = 1/2 \; (1-\mu^2)\, c$$

$$J_o = \left[-2(1-\mu^2)c - 2(1-\mu)c^2 - \frac{2t\lambda^2 H}{D} c^2 \right]$$

$$L_o = -\frac{2t\lambda^2 H}{D} c^4$$

$$M_o = \left[+3(1-\mu^2)c + 4(1-\mu)c^2 + c^4 + \frac{4t\lambda^2 H}{D} c^2(1+c^2) + \frac{k\lambda^4 v}{D} \right]$$

$$N_o = \left[-2(1+c^2) - \frac{K\lambda^2 H}{2t} c^2 \right] \qquad (12)$$

$$R_o = \left[-2 - 4c^2 + 2\mu c^2 - \frac{2t\lambda^2 H}{D} c2 \right]$$

$$U_o = \frac{2t\lambda^2 H}{D} c^2$$

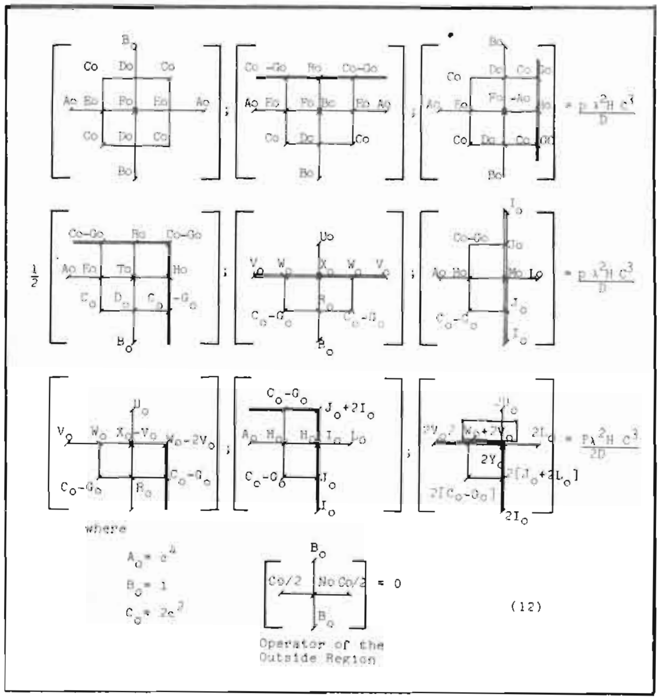
$$V_o = 1/2\; (1-\mu^2)\, c^3$$

$$W_o = \left[-2(1-\mu)c^2 - 2(1-\mu^2)c^3 - \frac{2t\lambda^2 H}{D} c^4 \right]$$

$$X_o = \left[1 + 4(1-\mu)c^2 + 3(1-\mu^2)c^3 + \frac{4t\lambda^2 H}{D} c^2(1+c^2) + \frac{k\lambda^4 v}{D} \right]$$

$$V_o = \left[2(1-\mu)c^2 + \tfrac{1}{2}(1-\mu^2)c^3 + \tfrac{1}{2}(1-\mu^2)c + \frac{4t\lambda^2 H}{D} c^2(1+c^2) + \frac{k\lambda^4 v}{D} \right]$$

$\lambda H, \lambda v$ = distance between node points of a network



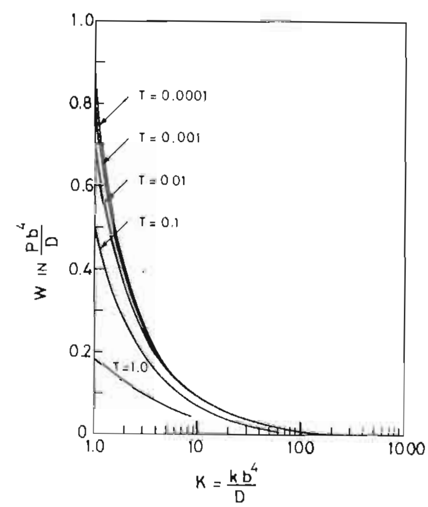


FIG. 1 EFFECT OF FLEXIBILITY INDEX ON THE DEFLECTION

7. Results

The deflections and moments of the plate are studied as functions of the main parameters that enter into the problem, namely, the flexibility index K, and the equivalent membrane tension modulus T of the subgrade.

Sixty problems were solved for a rectangular plate, resting freely on an elastic subgrade, and subjected to uniform and concentrated loads.

A wide range of parameters were considered for uniformly loaded plates because the effect of the equivalent membrane tension on moments and deflections is more pronounced for such loading. This is true because the moments in uniformly loaded plates resting freely on a subgrade without shearing forces is equal to zero and the deflections of such plates are uniform. Plates with concentrated loads were studied to find the effects of the parameters on more practical cases.

To determine the influence of K and T, six values of K ranging from 1.0 to 100,000 and five values of T in the range between 0.0001 and 1.0 are considered.

Typical graphs are shown in Figs. 1, 2, 3, 4,.

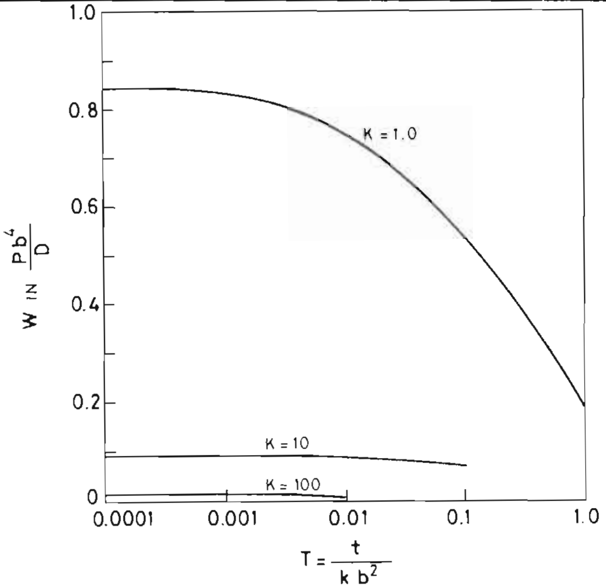


FIG. 2 EFFECT OF MEMBRANE TENSION ON THE DEFLECTION

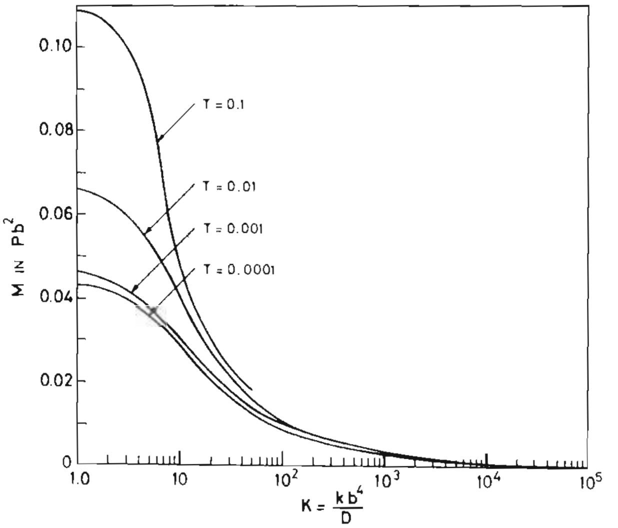


FIG. 3 EFFECT OF FLEXIBILITY INDEX ON THE MOMENT

8. Conclusions

On the basis of the numerical results obtained it is concluded that:

1) The larger the flexibility index K the smaller is the magnitude of the deflection w and the moment M or, in other words, for flexible plate and hard springs the deflections of the plate are reduced and the moments in the plate vanish.

2) As the membrane tension T increases the deflections of the plate decrease. However, the effect of the load is distributed to the region outside the plate, thus giving larger deflections outside the plate. For an extremely large tension force the deflections everywhere approach zero. It is clear, therefore, that the effect of the membrane, assumed to connect the vertical springs, is to cause the distribution of the load on the subsoil, bringing the problem closer to the actual behavior of soil.

3) Inspection of Fig. 4 reveals that as the membrane tension T increases the moments in the plate increase. This phenomenon should be expected if one recalls that for a membrane-free subgrade or, in other words, for a Winkler-type subgrade, the moments in a uniformly loaded plate, supported freely on the subgrade, are zero.

4) In changing the loading from distributed to concentrated at the middle, there is usually an increase in the maximum deflection for small values of the flexibility index K and the membrane tension T. However, a small increase in K or T has a pronounced effect in reducing the deflection of the plate. Moments are also decreased for larger values of the parameters.

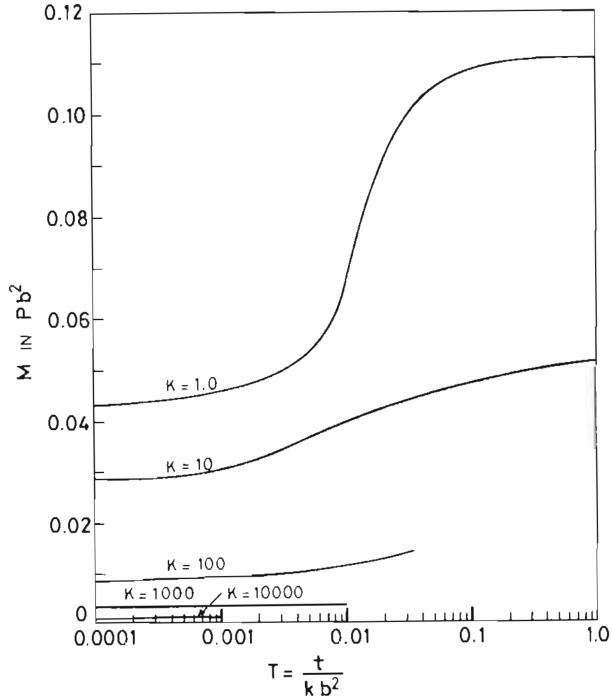


FIG. 4 EFFECT OF MEMBRANE TENSION ON THE MOMENT

REFERENCES

(1) *Winkler, E., Die Lehre von der Elastizitat und Festigkeit, Prag*, 1867, *p.* 182.

(2) *Westergaard, H.M., "Stresses in Concrete Pavements Computed by Theoretical Analysis," Public Roads, April*, 1926.

(3) *Westergaard, H. M. , "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," Public Roads, December*, 1933.

(4) *Westergaard, H. M. "Stresses in Concrete Runways of Airports," Proceedings, Highway Research Board, National Research Council, Vol.* 19, 1939.

(5) *Westergaard, H. M.., "New Formulas for Stresses in Concrete Pavements of Airfields," Transactions, ASCE, Vol* 113, 1948.

(6) *Schleicher, F., Kreisplatten auf elastischer Unterlag, Berlin*, 1926.

(7) *Timoshenko, S. and Woinowsky-Krieger, S., Theory of plates and Shells, McGraw-Hill Book Company, New York*, 1959.

(8) *Pickett, G., Raville, M.E., Janes, W. C. and McCormick, F.J., "Deflections, Moments and Reactive Pressures for Concrete Pavements," Bulletin No.65, Kansas State College, Manhattan, Kansas, October*, 1951.

(9) *Reissner, E., "Stresses in Elastic Plates Over Flexible Subgrades,"Transactions,ASCE,Vol.***122**, 1957.

(10) *Boussinesq, J., "Applications des Potentiels a l'Etude de l'Equilibre et du Mouvement des Solides Elastiques," Paris* 1885.

(11) *Terazawa, K., "On the Elastic Equilibrium of a Semi-Infinite Elastic Solid," Journal Coll. of Science, Imperial University of Tokyo, XXXVII, December* 1916.

(12) *Newmark, N.m., "Influence Charts for Computation of Stresses in Elastic Foundations," Univ. of Illinois Bull. No.* 338, 1942.

(13) *Newmark, N.M., "Influence Charts for Computation of Vertical Displacements in Elastic Foundations, "Univ. of Illinois Bull. No.* 367, 1947.

(14) *Marguerre, K., "Druckverteilung durch eine elastiche Schicht auf starrer raucher Unterlag," Ingenieur-Archiv*, 2, 1931, *pp.* 108-117.

(15) *Marguerre, K., "Spannungsverteilung and Well lenausbreitung in der kontinuierlich gestutzen Platte," Ingenieur-Archiv*, 4, 1933, *pp.* 332-353.

(16) *Biot, M.A., "Effect of Certain Discontinuities on the pressure Distribution in a Loaded Soil," Physics, Vol.* 6, *December*, 1935, *p.* 367.

(17) *Pickett, G., "Stress Distribution in a Loaded Soil with Some Rigid Boundaries," Proceedings, Highway Research Board, National Research Council, Vol.* 18, 1938, *pp.* 35-48.

(18) *Passer, W., "Druckverteilung durch eine elastische Schict," Akad der Wissenschaften in Wien, Math-Natur, Klasse Abt. IIa*, 144 *Band* 5 *and* 6, *Heft, Wien*, 1935.

(19) *Serebryanyi, R.V., "The Deflection of a Thin Semi-Infinite plate Resting on an Elastic Layer of Finite Thickness," Soviet Physics Doklady, Vol.* 4, -1-3, 1959 (*Translated from Russian*).

(20) *Hogg, A.H.A., "Equilibrium of a Thin Plate, Symmetrically Loaded, Resting on an Elastic Foundation of Infinite Depth," Philosophical Magazine, Series*, 7, *Vol.* 25, 1938, *pp.* 576-582.

(21) *Holl, D.L., "Thin Plates on Elastic Foundation, "Proceedings*, 5*th International Cong. for Applied Mechanics, Cambridge, Mass.*, 1938, *John Wiley and Sons, Inc., New York, N.Y.*, 1939, *pp.* 71-74.

(22) *Volterra, E., "On the General Problem of a Plate Supported by Elastic Soil," R.C. Accad, Lincei, May*, 1947, *Vol.* 2, *pp.* 596-598. (*In Italian*).

(23) *Burmister, D.M., "Theory of Stresses and Displacements in Layered Systems and Application to the Design of Airport Runways," Proceedings, Highway Research Board National Research Council, Vol.* 23, 1943, *pp.* 126-148.

(24) *Burmister, D.M., "The General Theory of Stresses and Displacements in Layered Systems," Journal of Applied Physics, Vol.* 16, 1945, *pp.* 89-94; 126-127; 296-302.

(25) *Burmister, D.M., "Stress and Displacement Characteristics of a Two-Layer Rigid Base System Influence Diagrams and Practical Applications," Proceedings, Highway Research Board, National Research Council, Vol.* 35, 1956.

(26) *Fox, L., "Computation of Traffic Stresses in a Simple Road Structure," Dept. of Scientific and Industrial Research, Road Research Laboratory, Technical Paper No.* 9. 1948.

(27) *Acum, W.E.A. and Fox, L., "Computation of Load Stresses in a Three Layer Elastic System," Geotechnique, Vol.LL, No.*4, *December*, 1951, *pp*. 293-300.

(28) *Mehta, M.R. and Veletsos, A.S., "Stresses and Displacements in Layered Systems," Univ. of Illinois Civil Engineering Structural Research Series No.* 178, *June*, 1959.

(29) *Lemcoe, M.M., "Stresses in Layered Elastic Solids," Transactions, ASCE, Vol.*126, *Part I*, 1961.

(30) *Vlasov, V.Z. and Leont'ev, N.N., "Beams, Plates and Shells on Elastic Foundation," Gos. Izd.Phys,-Mat. Lit., Moscow*, 1960. (*In Russian*).