# حول وجود مناطق اتزان في الاضطراب الهوائي الكهرومغناطيسي

سيد دسوقي حسن*

يظهر البحث وجود منطقتين للاتزان (مغناطيسياً وحركياً) في حالة كبر رقمي رينولد (المغناطيسي والحركي) وضعف المجال المغناطيسي .

وتعرف منطقة الاتزان على أنها مجموعة أرقام الموجات التي يتم فيها معظم فقدان الطاقة والتي تتميز احصائياً بأنها منتظمة مع الوقت وأيزوتروبية ولا تعتمد على حالة الطاقة في مناطق الاضطراب الكبير .

ولو كان رقم رينولد الحركي كبيراً بينما رقم رينولد المغناطيسي صغيراً والمجال المغناطيسي قوياً فلسوف يوجد في هذه الحالة تماثل محوري في نفس مجموعة أرقام الموجات .

* استاذ مشارك بقسم الهندسة الميكانيكية كلية الهندسة – جامعة الرياض

# On The Existence Of Equilibrium Ranges In Hydromagnetic Turbulence

**Sayed Desouki Hassan** *

*Two equilibrium ranges (magnetic and kinetic) are shown to exist if the two Reynolds numbers (magnetic and kinetic) are very large and the magnetic field is not very strong. An equilibrium range is defined to be a range of wave numbers which is responsible for most of the energy dissipation of the system and for which statistical conditions are steady, isotropic and independent of the conditions of the energy containing eddies. If the kinetic Reynolds number is large but the magnetic Reynold number is small and the magnetic field is strong then we will have only axisymmetry in the same range of wave numbers.*

**Nomenclature**

$\vec{h}$ = induced magnetic field

$\vec{H}$ = applied magnetic field

$\vec{j}$ = electric current density

$\vec{E}$ = electric field

$\vec{u}$ = velocity

$e$ = density

$\nu$ = kinematic viscosity

$\sigma$ = conductivity

$\mu$ = magnetic permeability

$\vec{W}, \vec{V}$ = Alven velocities

$\lambda$ = magnetic viscosity coefficient

$d\vec{Z}(\vec{k}), d\vec{M}(\vec{k})$ = Fourier coefficients of velocity and Alven velocity

F ( ) = Fourier transform of ( )

$\vec{k}$ = wave number vector

E ( ) = ensemble average of ( )

$\Phi_{ij}(\vec{k})$ = velocity correlation tensor

## 1. INTRODUCTION

The existence of equilibrium ranges in magnetohydrodynamic turbulence has been assumed by many authors. The present work is concerned with the question of existence of these ranges. Our method of attack is to analyze turbulence as a mechanical system (by using Fourier techniques) possessing an infinite number of degrees of freedom (corresponding to different wave numbers in Fourier analysis).

The question of existence can now be expressed in the following way: Does there exist a range of degrees of freedom (wave numbers) which is responsible for most of the energy dissipation of the system and for which statistical conditions are steady, isotropic and independent of the conditions of the energy containing these degrees of freedom (i.e., the range of wave numbers containing most of the energy)?

To answer this question one has to analyze the forces acting on the system and find their effect on energy distribution among different degrees of freedom, i.e., we have to analyze the flow of energy in the number space.

## 2. Fourier Analysis

There is a possible way of describing the turbulent field as a mechanical system possessing an infinite number of degrees of freedom, namely, the method of Fourier analysis.

* *Associate Professor, Department of Mechanical Engineering, College of Engineering - University of Riyadh*

Energy flows from one degree to another. The system is not conservative, since energy is dissipated by viscosity and resistivity of the fluid. In a stationary state, energy is supplied by external sources and the same energy flows out of the system by the dissipative effects as mentioned above.

$$\text{curl}\ \vec{E} = -\mu \frac{\partial \vec{h}}{\partial t} \quad \vec{j} = \sigma\,(\vec{E}+\mu\,\vec{u}\times(\vec{H}+\vec{h})) \qquad (1)$$

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\,\vec{u}\right] = \mu\,\vec{j}\times(\vec{H}+\vec{h}) + \nu\rho\nabla^{2}\vec{u} - \nabla p \qquad (2)$$

Introducing the two Alven velocities $\vec{W} = \vec{H}/\sqrt{\mu\,\rho}$ and $\vec{V} = \vec{h}/\sqrt{\mu\,\rho}$ and the magnetic viscosity coefficient $\lambda = (1/\mu\sigma)$ into the set of Eqs. (1) and (2) we get two equations governing the velocity u and the Alven velocity $\vec{V}$:

$$\frac{\partial V_i}{\partial t} = \frac{\partial}{\partial x_j}(u_i\,V_j - u_j\,V_i) + W_j\,\frac{\partial u_i}{\partial x_j} + \lambda \frac{\partial^2}{\partial x_j\,\partial x_j} V_i \qquad (3)$$

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{a x_j}(W_jV_j + V_i V_j - u_iu_j) - \frac{\partial}{\partial x_i}\left[\frac{P}{\rho} + W_jV_j + \frac{1}{2}V^2\right] + \nu\nabla^2 u_i \qquad (4)$$

We define the Fourier coefficients of velocity and Alven velocity by:

$$d\vec{z}\,(\vec{k}) = \frac{1}{(2\pi)^3}\int \vec{u}\,(\vec{x})e^{-i\vec{k}\,.\,\vec{x}} \prod_{j=1}^{3}\left[\frac{e^{-idk_jx_j} - 1)}{-ix_j}\right] d\vec{x} \qquad (5)$$

$$d\vec{M}\,(\vec{k}) = \frac{1}{(2\pi)^3}\int \vec{V}\,(\vec{x})\,e^{-i\vec{k}\,.\,\vec{x}} \prod_{j=1}^{3}\left[\frac{e^{-idk_jx} - 1}{ix_j}\right] d\vec{x}\ , \qquad (6)$$

(no summation over j in $dZ_jx_j$). Moments involving $d\vec{Z}(\vec{k})$ and $d\vec{M}(\vec{k})$ will have the following properties.

$$\overline{dZ_i\,(\vec{k}_1)\,dZ_j\,(\vec{k}_2)} = 0 \qquad \vec{k}_1 + \vec{k}_2 \neq 0$$

$$= O\,[\,(\vec{dk})\,]\quad \vec{k}_1 + \vec{k}_2 = 0\ ;\ \text{for any}\ \ i, j = 1, 2, 3, \qquad (7)$$

$$\overline{dZ_{i1}\,(\vec{k}_1)\,dZ_{i2}\,(\vec{k}_2)\ldots dZ_{n}\,(\vec{k}_n)} = 0$$

$$\text{if}\quad \vec{k}_1 + \vec{k}_2 + \ldots + \vec{k}_n \neq 0 \qquad (8)$$

$$\overline{dZ_i\,(\vec{k}_1)\,dZ_j\,(\vec{k}_2)\,dZ_e\,(\vec{k}_3)} = 0 \quad [\,(\vec{dk})^2\,]$$

$$\text{if}\quad \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \qquad (9)$$

and for higher moments it is the cumulants of the distribution that continue the series

$$\overline{dZ_i\,(\vec{k}_1)\,dZ_j\,(\vec{k}_2)\,dZ_\ell\,(\vec{k}_3)\,dZ_m\,(\vec{k}_4)} - \overline{\{dZ_i\,(\vec{k}_1)\,dZ_j\,(\vec{k}_2)\}\,\{dZ_\ell\,(\vec{k}_3)\,dZ_m\,(\vec{k}_4)\}} - \ldots = O\,[\,(dk)^3\,]\quad \vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0 \qquad (10)$$

If in Eq. (10) for example, the four wave numbers are such that

$\vec{k}_1 + \vec{k}_2 = 0$ and $\vec{k}_3 + \vec{k}_4 = 0$, then

$\overline{\{dZ_i(\vec{k_1})\, dZ_j(\vec{k_2})\}\{dZ(\vec{k_3})\, dZ_{m\varrho}(\vec{k_4})\}} = O[(dk)^2]$

will dominate $O[(dk)^3]$ and we have the interesting result

$$\overline{dZ_i(\vec{k}_1)\, dZ_j(\vec{k}_2)\, dZ_\varrho(\vec{k}_3) dZ_m(\vec{k}_4)} \rightarrow$$

$$\overline{\{dZ_i(\vec{k}_1)\, dZ_j(\vec{k}_2)\}\;\{dZ_\varrho(\vec{k}_3)\, dZ_m(\vec{k}_4)\}} \quad (11)$$

Moreover the conditions

$$\nabla.\vec{h} = \nabla.\vec{u} = 0$$

will lead to

$$k_i\, dZ_i(\vec{k}) = 0 \quad (12)$$

$$k_i\, dM_i(\vec{k}) = 0 \quad (13)$$

and from the definition of $d\vec{Z}(\vec{k})$ and $d\vec{M}(\vec{k})$ we have

$$d\vec{Z}^*(\vec{k}) = d\vec{Z}(-\vec{k}) \quad (14)$$

$$d\vec{M}^*(\vec{k}) = d\vec{M}(-\vec{k}) \quad (15)$$

where the asterisk * denotes conjugation.

we conclude this section by Fourier-analyzing Eqs. (3) and (4) where we obtain

$$\frac{\partial}{\partial t} dZ_i(\vec{k}) = i\left[(\delta_{mi} k_j - \frac{k_i k_j k_m}{k^2}\right]\int_{\vec{k'}} [dM_j(\vec{k} - \vec{k'})\, dM_m(\vec{k'}) - dZ_j(\vec{k}-\vec{k'})\, dZ_m(\vec{k'})]$$

$$+ ik_j W_j dM_i(\vec{k}) - \nu k^2 dZ_i(\vec{k}) \quad (16)$$

$$\frac{\partial}{\partial t} dM_i(\vec{k}) = i\int k_j\, [dM_j(\vec{k} - \vec{k'})\, dZ_i(\vec{k}) - dZ_j(\vec{k} - \vec{k'})\, dM_i(\vec{k'})] + ik_j W_j dZ_i(\vec{k}) - \lambda k^2 dM_i(\vec{k}) \quad (17)$$

where we used Parseval's theorem which states that

$$F(u_i u_j) = \int_{\vec{k'}} [F(u_i)]_{\vec{k}-\vec{k'}}\; [F(u_j)]_{\vec{k'}}$$

$$= \int_{\vec{k'}} dZ_i(\vec{k} - \vec{k'})\, dZ_j(\vec{k'}) \quad (18)$$

## 3. On the Direct and Indirect Interaction Theories

Before discussing the energy flow in the wave number space, we can make an interesting analysis of the relative order of magnitude of different terms that contribute to the rate of change of average quantities of the Fourier coefficients $d\vec{Z}(\vec{k})$ and $d\vec{M}(\vec{k})$.

The important result of this section is that the direct and indirect interaction terms (defined later on in this section) are of the same order of magnitude. The classical case was proved by Proudman [4]. Let us try to estimate the order of magnitude of the rate of change of statistical moments of the distribution $dZ(k)$, $dM(k)$.

$$\frac{\partial}{\partial t} E\{dZ_i(k)\, dZ_r(k_2)\, dM_\rho(k_3)\} = i\left[\delta_{mi} k_j - \frac{k_i k_j k_m}{k^2}\right]$$

$$\times \int_{\vec{k'}} E\, [dM_j(\vec{k}-\vec{k})\, dM_m(\vec{k})\, dZ_r(\vec{k_2})\, dM(\vec{k_3})$$

$$- dZ_j(\vec{k} - \vec{k'}) dZ_m(\vec{k'})\, dZ_r(\vec{k_2})\, dM\rho(\vec{k_3})]$$

$$- E[ik_j W_j dM_j(\vec{k})\, dZr(\vec{k_2})\, dM_\rho(\vec{k_3})]$$

$$- E(\nu k_2 dZ_i(\vec{k})\, dZ_r(\vec{k_2})\, dM(\vec{k_3}) \quad (19)$$

Referring to results of Eqs. (7) – (10) we have both sides of Eq. (19) equal to zero if $k+k_2+k_3 \neq 0$. However, if $k+k_2+k_3 = 0$, and if in the integrand of Eq. (19) we have $\vec{k'} = -\vec{k_2'}$ then $(\vec{k}-\vec{k'}) = -\vec{k_3}$. Similarly if $\vec{k'} = -\vec{k_3'}$ then $(\vec{k}-\vec{k'}) = -\vec{k_2}$; thus we have

$$\frac{\partial}{\partial t} E\{dZ_i(\vec{k})\, dZ_r(\vec{k_2})\, dM\rho(\vec{k_3})\}$$

$$= i\left[\delta_{mi}k_j - \frac{k_i k_j k_m}{k^2}\right]$$

$$\times\ E[dM_j(-\vec{k_2})dZ_r(\vec{k_2})]\ E[dM_m(-\vec{k_3})\,dM(\vec{k_3})]$$
$$+\ E[dM_j(-\vec{k_3})\,dM\rho(\vec{k_3})]\,E[dM_m(-\vec{k_2})\,dZ_r(\vec{k_2})]$$
$$-\ E[dZ_j(-\vec{k_3})\,dM\rho(\vec{k_3})]\,E[dZ_m(-\vec{k_2})\,dZ_r(\vec{k_2})]$$
$$-\ E[dZ_j(-\vec{k_2})\,dZ_r(\vec{k_2})]\ E[dZ_m(-\vec{k_3})\,dM_p(\vec{k_3})]$$

$$+\ i\left[\delta_{mi}k_j - \frac{k_i k_j k_m}{k^2}\right]\int_{\vec{k'}}[4^{th}\text{ cumulant of } dM_j(\vec{k}-\vec{k'})\,dM_m(\vec{k'})dZ_r(\vec{k_2})dM\rho(\vec{k_3})]$$

$$-\ i\left[\delta_{mi}k_j - \frac{k_i k_j k_m}{k^2}\right]\int_{\vec{k'}}[4^{th}\text{ cumulant of } dZ_j(\vec{k}-\vec{k'})\,dZ_m(\vec{k'})dZ_r(\vec{k_2})\,dM\rho(\vec{k_3})$$

$$-\ E[ik_jW_j dM_i(\vec{k})dZ_r(\vec{k_2})\,dM\rho(\vec{k_3})]$$
$$-\ E[\nu k^2\,dZ_i(\vec{k})dZ_r(\vec{k_2})dM\rho(\vec{k_3})] \qquad (20)$$

The first term on the right-hand side of Eq. (20) is $O[\vec{dk^2}]$, the second and third terms being integrals over an integrand of $O[(\vec{dk})^3]$ are of $O[(\vec{dk})^2]$, and the fourth and fifth terms are of $O[(\vec{dk})^2]$.

We notice that the first term on the right-hand side of Eq. (20) singles out the interaction of $\vec{k}$, $\vec{k_2}$ and $\vec{k_3}$ with themselves. These can be called "direct interaction terms".

The second and third terms on the right-hand side of Eq. (20) represent the interaction of $\vec{k}$, $\vec{k_2}$ and $\vec{k_3}$ with all other wave numbers and can be called the "indirect interaction terms".

Some classical theories of ordinary turbulence are based on neglecting the "indirect interaction terms" and are sometimes called "zero fourth-order cumulant" theories. But, as we have seen, there is a comparable contribution of the direct interaction terms and the indirect interaction terms. Thus one does not expect any good results from carrying out these theories in the case of hydromagnetic turbulence, unless there is some situation in which the indirect interaction terms can be neglected. However, this case is not obvious from our above analysis.

## 4. Energy Spectrum Equations

Here we want the governing equation of the quantity $dZ_m(\vec{k})dZ_i^*(\vec{k})$. This can be obtained from Eq. (16) by writing a similar equation for $dZ_m(\vec{k})$, conjugating Eq. (16), and then multiplying the equation for $dZ_i^*$ by $dZ_m$ and the one for $dZ_m$ by $dZ_i^*$. Adding these we get

$$\frac{\partial\, dZ_m(\vec{k})dZ_i^*(\vec{k})}{\partial t} = i\int_{\vec{k'}}[k_j dZ_j^*(\vec{k}-\vec{k'})dZ_i^*(\vec{k})\,dZ_m(\vec{k})$$
$$-k_j\,dZ_j(\vec{k}-\vec{k'})dZ_m(\vec{k'})dZ_i^*(\vec{k})]$$
$$+\ i\int_{\vec{k'}}[k_j dM_j(\vec{k}-\vec{k'})dM_m(\vec{k'})dZ_i(\vec{k})$$
$$-\ k_j dM_j(\vec{k}-\vec{k'})\,dM_i^*(\vec{k'})dZ_m(\vec{k})]$$
$$+\ i\int_{\vec{k'}}\frac{k_m}{k^2}dZ_i^*(\vec{k})\vec{k}.\ dz(\vec{k}-\vec{k'})\,dZ(\vec{k'})$$
$$-\ dM(\vec{k}-\vec{k'})\,dM(\vec{k'})\ .\ \vec{k}$$
$$-\ i\int_{\vec{k'}}\frac{\pi k_i}{k^2}dZ_m(\vec{k})\vec{k}.\ [dZ^*(\vec{k}-\vec{k'})\,dZ^*(\vec{k})$$
$$-\ dM^*(\vec{k}-\vec{k'})\,dM^*(\vec{k'})\ ]\ .\ \vec{k}$$
$$+\ ik_jW_j\,dM_m(\vec{k})\,dZ_i^*(\vec{k})$$
$$-\ dM_i^*(\vec{k})\,dZ_m(\vec{k}) - 2\nu k^2 dZ_m(\vec{k})dZ_i^*(\vec{k}) \qquad (21)$$

Similarly, by using Eq. (17) we obtain for $dM_i^*(\vec{k})dM_m(\vec{k})$:

$$\frac{\partial\, dM^*(\vec{k})\,dM_m(\vec{k})}{\partial t} = i\int_{\vec{k'}}[k_j dM_j(\vec{k}-\vec{k})\,dZ_m(\vec{k})\,dM_i^*(\vec{k}) - k_j dM_j^*(\vec{k}-\vec{k})dZ_i^*(\vec{k})dM_m(\vec{k})\ ]$$

$$+ i \int_{\vec{k}} [ k_j dZ_j^*(\vec{k}-\vec{k}) dM_i^*(\vec{k}) \, dM_m(\vec{k})$$

$$- k_j dZ_j(\vec{k}-\vec{k}) \, dM_m(\vec{k}) \, dM_i^*(\vec{k})]$$

$$+ ik_j W_j \, [dZ_m(\vec{k}) dM_i^*(\vec{k})$$

$$- dZ_i^*(\vec{k}) dM_m(\vec{k})] - 2\lambda k^2 dM_m(\vec{k}) \, dM_i^*(\vec{k}) \quad (22)$$

Now we will show the roles of the different types of terms on the right-hand side of Eqs. (21) and (22) in the flow of energy in the wave number space.

## 5. The Role of Inertial Forces

The contribution of inertial forces to $\frac{\partial}{\partial t} \int dZ_m(\vec{k}) \, dZ_i^*(\vec{k})$ is given by

$$\left[ \frac{\partial}{\partial t} dZ_m(\vec{k}) dZ_i^*(\vec{k}) \right]_{\substack{\text{inertial} \\ \text{forces}}}$$

$$= i \int_k \int_{k'} [k_j dZ_j^*(\vec{k}-\vec{k}) dZ_i^*(\vec{k}) dZ_m(k)$$

$$- k_j dZ_j(\vec{k}-\vec{k}) dZ_m(\vec{k}) dZ_i^*(\vec{k})] \quad (23)$$

Using the relation $\overrightarrow{dZ^*(k)} = dZ(-\vec{k})$, the right-hand side of Eq. (23) becomes

$$i \int_k \int_{k'} [ k_j dZ_j(\vec{k'}-\vec{k}) dZ_i(-\vec{k'}) dZ_m(\vec{k}) - k_j dZ_j(\vec{k}-\vec{k}) dZ_m(\vec{k'}) dZ_i(-\vec{k})]$$

By interchanging $\vec{k}, \vec{k'}$ in the first term in the integrand, it becomes

$$k_j dZ_j(\vec{k}-\vec{k'}) \, dZ_i(-\vec{k}) \, dZ_m(\vec{k'}) -$$
$$k_j dZ_i(\vec{k}-\vec{k'}) \, dZ_i(-\vec{k}) \, dZ_m(\vec{k'})$$

Using the relation $(k_j - k_j) dZ_j(\vec{k}-\vec{k'}) = 0$, we then have :

$$\left[ \frac{\partial}{\partial t} dZ_m(\vec{k}) \, dZ_i^*(\vec{k}) \right]_{\text{inertial forces}} = 0 \quad (24)$$

This shows that the role of inertial forces in hydromagnetic turbulence is the same as in classical turbulence. The role of inertial forces is to transfer kinetic energy from one part of the spectrum to another, but not to change the total amount of energy associated with a particular directional component of energy.

## 6. The Role of Pressure Forces

Contracting (m) and (i) in Eq. (21) and using the continuity relation $k_i dZ_i(\vec{k}) = k_i dZ_i^*(\vec{k}) = 0$, the contribution of the mechanical and magnetic pressure (third and fourth terms on the right-hand side of Eq. (21) to the rate of change of $dZ_i(\vec{k}) dZ_i^*(\vec{k})$ is zero, i.e.

$$\left[ \frac{\partial}{\partial t} dZ_i(\vec{k}) dZ_i^*(\vec{k}) \right]_{\text{pressure forces}} = 0 \quad (25)$$

Thus the effect of pressure forces is to transfer energy from one directional component of dZ (k) to another in such a way as to conserve the total energy contributed by any small region of wave number space.

## 7. The Role of the Magnetic Energy Convective Term

This is the second term on the right-hand side of Eq. (22) which arises from the term $u_j(\partial V_j/\partial x_j) V_j$ in the induction equation (3), representing the rate of change of $\vec{V}$ due to convection. Let us contract (m) and (i) in Eq. (22) and integrate over $\vec{k}$. We notice first that

$$\int_{\vec{k'}} \int_{\vec{k}} k_j dZ_j^*(\vec{k}-\vec{k'}) dM_i^*(\vec{k'}) \, dM_i(\vec{k})$$

$$= \int_{\vec{k'}} \int_{\vec{k}} k_j dZ_j(\vec{k'}-\vec{k}) dM_i(-\vec{k'}) dM_i(\vec{k})$$

$$= \int_{\vec{k}} \int_{\vec{k'}} k_j dZ_j(\vec{k}-\vec{k'}) dM_i(-\vec{k}) dM_i(\vec{k'})$$

$$= \int_{\vec{k}} \int_{\vec{k'}} k_j dZ_j(\vec{k}-\vec{k'}) dM_i(-\vec{k}) \, dM_i(\vec{k'})$$

where the different steps have been achieved by using first Eqs. (14) and (15), then interchanging k and k' and finally, by using Eq. (12).

$$\frac{\partial}{\partial t} \int_{\vec{k}} dM_i^*(\vec{k}) \, dM_i(\vec{k}) = 0 \quad \begin{matrix}\text{convective} \\ \text{term}\end{matrix} \quad (26)$$

we conclude from Eq. (26) that the convective term will change the total energy associated with certain directions (for example, $\int_{\vec{k}} dM_i^*(\vec{k})\, dM_j(\vec{k})$, $(i \neq j)$ in such a way as to conserve the total energy of the magnetic system.

## 8. Leakage of Energy from the Mechanical System to the Magnetic System and Vice Versa

This leakage of energy between the two systems is represented by the second and fifth terms on the right-hand side of Eq. (21) and the first and third terms on the right-hand side of Eq. (22). These terms arise from the terms $[V_j(\partial u_i/\partial x_j) + W_j\,(\partial u/\partial x_j)]$ in Eq. (3), and $[(\partial\,(W_j V_i + V_i V_j)\,/\partial x_j)]$ in Eq. (4).

Physically these terms represent the tendency of the turbulent motion to stretch the lines of force, thereby increasing the magnetic energy, but the lines of force will then tend to contract and in so doing they will accelerate the fluid, thus increasing the kinetic energy. We will show in this section, that the contribution of these terms to the rate of change of the total amount of energy, i.e., magnetic and kinetic energy, is zero.

Consider the second term on the right-hand side of Eq. (21). After contracting (m) and (i), integrating over $\vec{k}$ and replacing $dZ_i^*(\vec{k})$ by $dZ_i(-\vec{k})$ and $dM_i^*(\vec{k'})$ we obtain

$$\int_{\vec{k}}\int_{\vec{k'}} [k_j dM_j(\vec{k}-\vec{k'})\, dZ_i(-\vec{k}) dM_i(\vec{k'})$$

$$k_j dM_j(\vec{k'}-\vec{k}) dZ_i(\vec{k})\, dM_i(-\vec{k'})]$$

$$= \int_{\vec{k'}}\int_{\vec{k}} [k_j dM_j(\vec{k'}-\vec{k}) dZ_i(-\vec{k'}) dM_i(\vec{k})$$

$$- k_j dM_j(\vec{k'}-\vec{k}) dZ_i(\vec{k'}) dM_i(-\vec{k})$$

$$= \int_{\vec{k'}}\int_{\vec{k}} [k_j dM_j(\vec{k'}-\vec{k}) dZ_i(-\vec{k'}) dM_i(\vec{k})$$

$$- k_j dM_j(\vec{k'}-\vec{k}) dZ_i(\vec{k'})\, dM_i(-\vec{k})]$$

which is equal to the first term on the right-hand side of Eq. (22) with the minus sign.

Thus, the contribution of these stretching terms (second and fifth terms on the right- hand side of Eq. (21) and first and third terms on the right-hand side of Eq. (22) to the rate of change of the quantity

$$\int_{\vec{k}} [dZ_i(\vec{k})\, dZ_i^*(\vec{k}) + dM_i(\vec{k})\, dM_i^*(\vec{k})]$$

is zero.

## 9. The Energy Equation for the Kinetic System

We recall first the relation between the Fourier transform of the velocity correlation tensor $\phi_{ij}(\vec{k})$ and $dZ(\vec{k})$ which is given by

$$\phi_{ij}(\vec{k}) = \lim_{\vec{dk} \to 0} \frac{dZ_i(\vec{k})\, dZ_j^*(\vec{k})}{\vec{dk}} \qquad (27)$$

$\phi_{ij}(\vec{k})$ is given by:

$$\phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int R_{ij}(\vec{r})\, e^{-i\vec{k}.\vec{r}}\, dr \qquad (28)$$

where $R_{ij}(\vec{r}) = \overline{u_i(\vec{x})\, u_j(\vec{x}+\vec{r})}$

Thus

$$\frac{1}{2}\overline{u_i(\vec{x})\, u_i(\vec{x})} = \frac{1}{2}\int \phi_{ii}(\vec{k})\, \vec{dk} \qquad (29)$$

which represents the kinetic energy per unit mass. Therefore, contracting m, i in Eq. (21) and integrating it over all $\vec{k}$ we get:

$$\frac{\partial}{\partial t}\frac{1}{2}\int \phi_{ii}(\vec{k})\, \vec{dk} = \int I_{MK}\, \vec{dk} - \nu \int k^2 \phi_{ii}(\vec{k})\, dk \qquad (30)$$

when $I_{MK}$ represents the interaction term between the magnetic system and the kinetic system which we analyzed in Section1 .7. Actually, $\Omega_{ii} = k^2 \phi_{ii}(\vec{k})$ represents the spectrum of the vorticity correlation $\overline{w_i(x)\, w_i(x)}$ as can be shown easily.

Notice that viscous forces are more significant for small eddies than for large eddies and represent the only energy sink in absence of magnetic interaction. However, in the presence of magnetic forces we have another sink (or source) represented by $I_{MK}(k)$ which is given by:

$$I_{MK} = \frac{i}{2} \int \lim_{\vec{k'}\vec{dk} \to 0} \frac{1}{\vec{dk}} \Big[ k_j dM_j(\vec{k}-\vec{k'}) dM_i(\vec{k'}) dZ_i^{*}(\vec{k}) - k_j dM_j^{*}(\vec{k}-\vec{k'})\, dM_i^{*}(\vec{k'})\, dZ_i(\vec{k}) \Big]$$

$$+\frac{i}{2} W_j \lim_{\vec{dk}\to 0} \frac{1}{\vec{dk}} \Big[ (k_j)\, dM_i(\vec{k}) dZ_i^{*}(\vec{k}) - dM_i^{*}(\vec{k}) dZ_i(\vec{k}) \Big] \tag{31}$$

## 10. The Energy Equation for the Magnetic System

First we define:

$$\Gamma_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int H_{ij}(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, dr, \tag{32}$$

where

$$H_{ij}(\vec{r}) = \overline{V_i(\vec{x})\, V_j(\vec{x}+\vec{r})} \tag{33}$$

as the magnetic field correlation tensor.

Also,

$$\Gamma_{ij}(\vec{k}) = \lim_{\vec{dk}\to 0} \frac{dM_i(\vec{k})\, dM_j(\vec{k})}{\vec{dk}} \tag{34}$$

and

$$\frac{1}{2}\overline{V_i(\vec{x}) V_j(\vec{x})} = \frac{1}{2} \int \Gamma_{ij}(\vec{k})\, \vec{dk} \tag{35}$$

Thus contracting m and i in Eq. (22) and integrating over k we get

$$\frac{\partial}{\partial t} \frac{1}{2} \int \Gamma_{ii}(\vec{k})\, \vec{dk} = \int I_{MK}\, \vec{dk} - \lambda \int k^2\, \Gamma_{ii}(k)\, dk \tag{36}$$

Therefore, energy is dissipated from the magnetic system by the resistivity of the fluid λ, which converts magnetic energy into heat, and by the interaction term $\int I_{MK}\, dk$, which converts magnetic energy into kinetic energy, and vice versa.

## 11. The Energy Equation for theTotal System

If we now consider the magnetic system and the kinetic system as one system, the energy equation of that system will be

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i u_i} + \frac{1}{2}\overline{V_i V_i}\right) = -\nu \int k^2 \phi_{ii}(\vec{k}) \vec{dk} - \lambda \int k^2\, \Gamma_{ii}(\vec{k})\, \vec{dk} \tag{37}$$

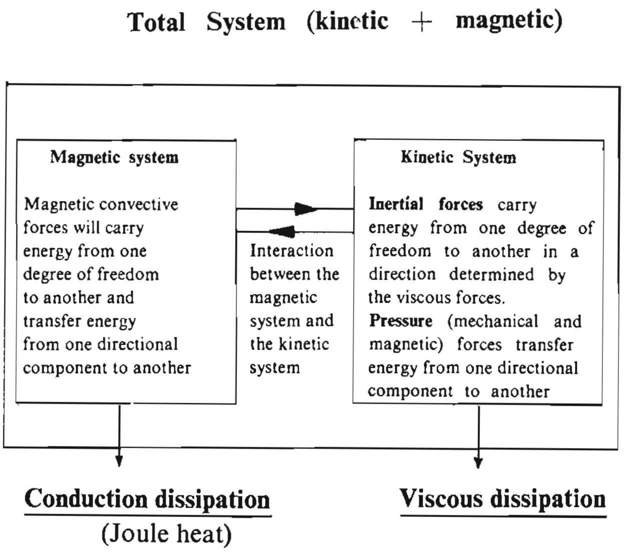
Thus energy flows out of the total system in the form of heat by the two dissipative agents; viscosity and resistivity.

## 12. The Flow Diagram

We have two energy bearing vectors, the velocity field and the magnetic field. Thus, in the terminology of statistical mechanics, we have two mechanical systems, the kinetic system and the magnetic system. Both systems are out of equilibrium. The kinetic system is acted upon by viscous forces (converting kinetic energy into heat), magnetic forces ( converting kinetic energy into magnetic energy and vice versa), and by inertial forces which carry energy from one degree of freedom to another and distribute energy among different directional components.

The magnetic system is acted upon by conduction forces (converting magnetic energy into Joule heat), interaction forces(converting magnetic energy into kinetic energy and vice versa), and by inertial forces which carry energy from one degree of freedom to another and distribute energy among different directional components.

One can depict the above conclusion in the following diagram:




### 13. On the Existence of the Magnetic Equilibrium Range

First we clarify what we mean by the magnetic equilibrium range. This is a range of large wave numbers which is responsible for most of the energy dissipation, and for which the magnetic field Fourier coefficients are statistically steady, isotropic, and independent of the Fourier coefficients of the range of wave numbers containing most of the energy.

Second, let us recall the results we obtained for the magnetic system. We proved in Section 7 that:

$$\left[\frac{\partial}{\partial t}\int_{\vec{k}} dM_i^*(\vec{k})\, dM_m(\vec{k})\right]_{\substack{\text{convective}\\ \text{terms}}} \begin{array}{ll} = 0 & \text{if} \quad i = m = 0 \\ \neq 0 & \text{otherwise (in general)} \end{array}$$

We also proved that convective forces are conservative, i.e., they do not carry energy out of the magnetic system. We conclude then that convective forces will try to redistribute the magnetic energy in different directions in such a way as to keep the total magnetic energy constant, and therefore will try to lead to isotropy.

In the presence of the external magnetic field $\vec{H}$, we recall that

$$\left[\frac{\partial}{\partial t}\int_{\vec{k}} dM_i^*(\vec{k})\, dM_m(\vec{k})\right]_{\vec{H}}$$

$$= ik_jW_j\,[\,dZ_m(\vec{k})dM_i^*(\vec{k}) - dZ_i^* dM_m(\vec{k})\,]$$

$$\begin{array}{ll} = 0 & m = i \\ \neq 0 & \text{otherwise} \end{array}$$

Also, we know that this term occurs with a negative sign in the kinetic spectrum Eq. (21) and therefore it represents the modulation of different directional components of the energy (by converting magnetic energy into kinetic energy and vice versa), while keeping the total amount of energy (kinetic or magnetic) constant.

Furthermore, we see from Eq. (22) that

$$\left[\frac{\partial}{\partial t} dM_i^*(\vec{k})\, dM_m(\vec{k})\right]_{\vec{H}} = 0$$

$$\text{if} \quad \vec{k}\,.\,\vec{W} = 0$$

i.e., eddies with wave numbers perpendicular to the external magnetic field are not affected by its presence. Also, we note that large eddies ($\vec{k} \to 0$) are not as much affected as the small eddies ($\vec{k} \to \infty$) by the external magnetic field. Thus, the motion of the small eddies will become axisymmetric around the external magnetic field $\vec{H}_o$.

We have, therefore, two competing types of forces, the external magnetic field which will try to lead to axisymmetry, and convective forces which will try to lead to isotropy. If the convective forces are dominating (as in the case when $R_m >> 1$) then isotropy is expected. However, if the external magnetic field is strong and convective forces are weak ($R_m << 1$), then one expects axisymmetry of the motion of the small eddies (large eddies are not affected as much).

In the early stages of the generation of the turbulence field, large eddies are created first (i.e., degrees of freedom corresponding to small wave numbers are excited first). The convection and the interaction of these large eddies (with each other) will create smaller eddies. In other words, energy is transferred to larger wave numbers. The energy received by a particular degree of freedom can either be dissipated by the action of conductivity, or transferred to higher wave numbers, or transformed into kinetic energy. If, for example, the magnetic Reynolds number is very large, dissipation will triumph over transfer only at very high wave numbers. As we have seen, in the process of transfer of energy, the convective forces will try to weaken the influence of external large-scale conditions of the motion by trying to eliminate directional preferences of the energy . Thus the motion associated with sufficiently large wave numbers should be isotropic. This statistical independence of the motion associated with large wave numbers from the motion associated with small wave numbers will lead to the conclusion that the former motion is in statistical equilibrium since no time-dependence can be imposed on the motion of the large wave numbers. The above argument, however, only points out the existence of the magnetic equilibrium range, and is far from being an analytical proof for such an existence.

**14. On the Existence of the Kinetic Equilibrium Range**

First, we clarify what we mean by the kinetic equilibrium range. This is a range of wave numbers which is responsible for most of the viscous dissipation and for which the velocity Fourier coefficients are statistically steady, isotropic , and independent of the Fourier coefficients of the range of wave numbers containing most of the kinetic energy.

Second, let us recall the results we obtained for the kinetic system. We showed that:

$$\left[ \frac{\partial}{\partial t} \int dZ_i \overrightarrow{(k)}\, dZ^*_m \overrightarrow{(k)} \right]_{\text{inertial forces}} = 0$$

$$\left[ \frac{\partial}{\partial t} dZ_i \, dZ^*_i \overrightarrow{(k)} \right]_{\text{pressure forces}} = 0$$

Thus, the role of inertial and pressure forces in the kinetic system is the same as that of the convective forces in the magnetic system. Also, we notice that the term corresponding to the external magnetic field in the magnetic system governing Eq.(22) occurs with the opposite sign in the equation governing the kinetic system, Eq. (21).

We can conclude then, by an argument similar to that of Section 13, that if the Reynolds number is very high and the external magnetic field is weak, isotropy is expected (and also the kinetic equilibrium range), while if the Reynolds number is small and the external magnetic field is strong, only axisymmetry of the motion of small eddies is expected.

**15. Summary**

In this paper we have studied the direct and indirect interaction theories and showed that in hydromagnetic turbulence that the approximation involved in these theories is not legitimate in general. We also studied the existence of different equilibrium ranges and showed that they do exist in hydromagnetic turbulence.

Finally we showed the flow of energy diagram.

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