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On The Vibrations of Elastic Plates

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The modes of vibration of a thin elastic plate are investigated for the two well-known conditions: the clamped and the simply-supported plate. Some relations are obtained between the two sets of eigenfunctions.

In the absence of external forces, and under suitable assumptions (see, for example, [1]), the lateral deflection of a thin elastic plate which lies in the xy-plane is governed by the linear equation

$$\Delta^2 w+\frac{\rho}{F}\frac{\partial^2 w}{\partial t^2}=0, \qquad (1.1)$$

where $w=w(x,y,t)$ is the deflection perpendicular to the xy-plane, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $\Delta^2=\Delta\Delta$, t is the time variable, ρ is the mass density per unit area of the plate and F is the flexural rigidity defined by

$$F=Eh^3/12(1-\nu^2)>0.$$

Here E is Young's modulus, h is the plate thickness and ν is Poisson's ratio.

In order to obtain the free vibrations of the plate, we assume that the motion is harmonic in time with frequency ω, *i.e.*

$$w(x,y,t)=u(x,y)f(t) \quad \text{and} \quad \frac{d^2f}{dt^2}+\omega^2 f=0.$$

This, together with eq. (1.1), imply

$$\Delta^2 u=\lambda u, \qquad (1.2)$$

where $\lambda=\rho\omega^2/F>0$.

Equation (1.2), which gives the free vibrations of the plate as the eigenfunctions of the operator Δ^2, has been studied extensively for various boundary conditions and has been completely solved for certain shapes of the plate. In particular, the rectangular plate which is simply supported at its edges, *i.e.* for which u satisfies the boundary condition

$$u=\frac{\partial^2 u}{\partial n^2}=0,$$

where n is the normal to the boundary, offers no difficulty. The circular plate which is clamped at its boundary, *i.e.* for which

$$u=\frac{\partial u}{\partial n}=0$$

on the boundary, can also be solved by separation of variables (see [2]). In the first case the eigenfunctions are sine functions and in the second case they are certain combinations of Bessel functions. However, the clamped rectangular plate and the simply supported circular plate have no solutions in terms of known functions. It is always possible, of course, to map one region conformally onto the other and attempt to solve the image of eq. (1.2) in the desired region. But the complexity of the resulting equation offers little hope of yielding anything more than a numerical approximation.

Since, under suitable assumptions, the eigenvalue equation (1.2) is known to have a complete set of orthogonal eigenfunctions, the difficulty that we face would seem to be a limitation on the method of separation of variables. In this state of affairs, where the eigenfunctions are known for one boundary and unknown for another, it may be interesting to investigate the relation between the two sets of eigenfunctions. It turns out, not surprisingly, that this relation is governed by the relation between the corresponding Green's functions, and that the Green's functions differ one from the other by certain reproducing kernels.

Existence and Properties of the Eigenfunctions

Let D be a finite open region in R^2 with a piecewise smooth boundary ∂D. For any two functions u and v defined on $\bar{D}=D\cup\partial D$ we obtain by two successive applications of Green's formula

$$\iint_D u\Delta^2 v\,dxdy = \iint_D \Delta u \Delta v\,dxdy + \int_{\partial D} \left(u\frac{\partial}{\partial n}\Delta v - \frac{\partial u}{\partial n}\Delta v\right) ds, \tag{2.1}$$

where n is the outward normal to ∂D and s is the variable of length along ∂D, provided these integrals exist. Let u be a solution of (1.2) which is three times differentiable in D. Then (*)

$$\iint u\Delta^2 u\,dxdy = \iint (\Delta u)^2 dxdy + \int \left(u\frac{\partial}{\partial n}\Delta u - \frac{\partial u}{\partial n}\Delta u\right) ds.$$

In order that the operator Δ^2 be symmetric we shall assume that

$$u\frac{\partial}{\partial n}\Delta u - \frac{\partial u}{\partial n}\Delta u = 0 \quad \text{on } \partial D,$$

and if the boundary conditions on u are to be homogeneous then we have four possibilities on ∂D:

(i) $u = \frac{\partial u}{\partial n} = 0$

(ii) $u = \Delta u = 0$

(iii) $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$

(iv) $\Delta u = \frac{\partial}{\partial n}\Delta u = 0.$

As we have already mentioned, the first and second boundary conditions describe a clamped and a simply supported edge, respectively. The third boundary condition describes an elastically supported edge and the fourth a free edge. Under boundary conditions (i) and (ii) it is evident that Δ^2 is positive definite, since it is positive and

$$\iint u\Delta^2 u\,dxdy = \iint (\Delta u)^2 dxdy$$
$$= 0$$
$$\Longrightarrow \Delta u = 0 \text{ in } D$$
$$\Longrightarrow u = 0 \text{ in D when } u = 0 \text{ on } \partial D$$

by the maximum principle for harmonic functions. We shall now show that we need only consider boundary conditions (i) and (ii), since (iii) and (iv) do not offer any significant additions to the eigenfunctions of (1.2).

Since

$$\lambda\iint u^2 dxdy = \iint u\Delta^2 u\,dxdy$$
$$= \iint (\Delta u)^2 dxdy$$
$$\geqq 0$$

we may assume without loss of generality that $\lambda = k^4$, where k is a real number. Eq. (1.2) is then equivalent to the pair of second order equations

$$\Delta u = k^2 v \tag{2.2}$$

$$\Delta v = k^2 u \tag{2.3}$$

in which u and v appear symmetrically. Since boundary condition (iv) may be expressed as $v = \frac{\partial v}{\partial n} = 0$ on ∂D for the differential equation $\Delta^2 v = k^4 v = \lambda v$ we conclude that every non-zero eigenvalue under boundary condition (i) corresponding to the eigenfunction u is also an eigenvalue under boundary condition (iv) corresponding to the eigenfunction $v = k^{-2}\Delta u$. For the case when $\lambda = 0$ it is not difficult to see that boundary condition (iv) implies u is an arbitrary harmonic function in D.

In order to see the significance of condition (iii) we rewrite equation (1.2) in the form

$$(\Delta - k^2)(\Delta + k^2)u = 0$$

which is equivalent to the pair of equations

$$(\Delta + k^2)u = v \tag{2.4}$$

$$(\Delta - k^2)v = 0. \tag{2.5}$$

*Double and single integrals are henceforth to be taken over D and ∂D respectively, unless otherwise specified.

Condition (ii) implies that $v=0$ on ∂D while (iii) implies that $\frac{\partial v}{\partial n}=0$ on ∂D. From Green's formula

$$\iint v\Delta v dxdy+\iint|\nabla v|^2 dxdy=\int v\frac{\partial v}{\partial n}ds,$$

and we conclude that equation (2.5) with either of these two conditions gives

$$k^2\iint v^2 dxdy+\iint|\nabla v|^2 dxdy=0$$

$\Longrightarrow kv=\nabla v=0$ in D

$k\neq 0\Longrightarrow v=0$ in D

$k=0\Longrightarrow v=$ constant in D.

If (ii) holds then $v=0$ on $\partial D\Longrightarrow v=0$ in D.

If (iii) holds then $\lambda=0$ is an eigenvalue of (1.2) corresponding to the solutions of $u=$constant in D.

Thus for $\lambda>0$ equation (1.2) under either boundary condition (ii) or (iii) reduces to equation (2.4) with $v=0$, *i.e.*

$$(\Delta+K^2)\quad u=0 \tag{2.6}$$

which is the well known equation for a vibrating membrane. Furthermore condition (ii) becomes $u=0$ on ∂D, and (iii) becomes $\frac{\partial u}{\partial n}=0$ on ∂D. These two conditions are special cases of the more general boundary condition for the membrane $\frac{\partial u}{\partial n}+\sigma u=0$, which is discussed in [2], corresponding to $\sigma=\infty$ and $\sigma=0$, and for which there exists a complete set of eigenfunctions and a corresponding increasing sequence of eigenvalues. Since each eigenvalue increases as σ increaes, the *nth* eigenvalue under condition (ii) is equal to or greater than the *nth* eigenvalue under condition (iii). In either case the plate is seen to have the same eigenfunctions as the membrane, with each eigenvalue for the plate equal to the square of the corresponding eigenvalue for the membrane.

In the special case when D is the rectangle $\{(x,y)|0<x<a,0<y<b\}$ the eigenfunctions for the simply supported plate are

$$u_{\nu\mu}(x,y)=\sin\frac{\nu\pi}{a}x\sin\frac{\mu\pi}{b}y \qquad \nu=1,2,3,...;\mu=1,2,3,...$$

with corresponding eigenvalues

$$\lambda_{\nu\mu}=(\frac{\nu^2}{a^2}+\frac{\mu^2}{b^2})\pi.$$

Clearly, an investigation of the vibrations of a plate is essentially a study of equation (1.2) under boundary conditions (i) or (ii), to which we now devote our attention. There are, of course, the special cases of mixed boundary conditions for certain shapes of the plate, but we shall not get involved in these technical problems.

The Clamped and the Simply Supported Plates

Let $\Omega=C^2(D)\cap L^2(D)\cap C(\bar{D})$ be the set of twice differentiable functions which are square integrable on D and continuous in $\bar{D}$. We then have the usual inner product.

$$(\varphi,\psi)=\iint\varphi\psi dxdy$$

for any pair of functions $\varphi,\psi\in\Omega$ and the norm

$$\|\varphi\|=\sqrt{(\varphi,\varphi)}.$$

In order to simplify the notation we shall use the complex variable $z=x+iy$ to denote points in $\bar{D}$. Let $g(z,z')$ be the Green's function for Laplace's equation in D, *i.e.* $g(z,z')+\frac{1}{2\pi}\log|z-z'|$ is harmonic and symmetric in both variables and $g=0$ on ∂D. For any $\varphi\in\Omega$

$$u(z)=(g,\varphi)=\iint g(z,z')\varphi(z')dx'dy'$$

is a well defined function in $C^4(D)\cap C^3(\bar{D})$ which vanishes on ∂D. We define the operator G on Ω by

$$G\varphi=(g,\varphi).$$

G is an inverse of $-\Delta$ in the sense that

$$\Delta G\varphi=-\varphi \text{ for any } \varphi\in\Omega \tag{3.1a}$$

$$G\Delta\varphi=-\varphi \text{ for any } \varphi\in\Omega \text{ which vanishes on } \partial D \tag{3.1b}$$

Let $W=\{u=G\varphi|\varphi\in\Omega\}$ and for any pair $u,v\in W$ we define the inner product

$$<u,v>=(\Delta u,\Delta v),$$

which is well defined since $\Delta u,\Delta v\in L^2(D)$. Note that

$\subset \Omega$, and that every function in W vanishes on ∂D. Let

$$|||u||| = \sqrt{<u,u>}$$

clearly $|||u||| \geqq 0$ and $|||u||| = 0 \Rightarrow u$ is harmonic in D and vanishes on $\partial D \Rightarrow u = 0$ by the maximum principle for harmonic functions

We seek the eigenfunctions of the operator Δ^2 in the set W which satisfy either of the two boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{(clamped plate)} \qquad (3.2a)$$

$$\Delta u = 0 \quad \text{(simply supported plate)} \qquad (3.2b)$$

The function

$$S(z,z') = \frac{1}{8\pi}|z-z'|^2 \log|z-z'|$$

is the fundamental biharmonic singularity which plays the same role for the operator Δ^2 as $-\frac{1}{2\pi}\log|z-z'|$ plays for Δ, *i.e.* S satisfies the biharmonic equation $\Delta^2 S = 0$ except at $z = z'$ and (*)

$$u(z) = \int (u\frac{\partial}{\partial n'}\Delta' S - \frac{\partial u}{\partial n'}\Delta' S - S\frac{\partial}{\partial n'}\Delta u + \frac{\partial S}{\partial n'}\Delta' u)ds'$$

for any biharmonic function u. Let $\Gamma(z,z')$ and $\gamma(z,z')$ be the Green's functions for the biharmonic equation under the boundary conditions (3.2a) and (3.2b) respectively, *i.e.* $\Gamma - S$ and $\gamma - S$ are biharmonic in D, $\Gamma = \frac{\partial \Gamma}{\partial n} = 0$ on ∂D, and $\gamma = \Delta\gamma = 0$ on ∂D in each of the variables z and z'.

Let $u \in W$ satisfy

$$\Delta^2 u = \lambda u \text{ in D.} \qquad (3.3)$$

from Eq. (2.1)

$$\iint (u\Delta'^2 v - v\Delta'^2 u)dxdy = \int (u\frac{\partial}{\partial n'}, \Delta' v - \frac{\partial u}{\partial n'}\Delta' v - v\frac{\partial}{\partial n'}\Delta' u + \frac{\partial v}{\partial n'}\Delta' u)ds'. \qquad (3.4)$$

If we replace $v(z')$ by $\Gamma(z,z')$ and D by $D - \{z' \big| |z'-z| < \varepsilon\}$, take the limit as $\varepsilon \to 0$, and use the properties of u and Γ we obtain

$$u(z) = \lambda \iint \Gamma(z,z')u(z')dx'dy' - \int \Delta'(z,z')\frac{\partial}{\partial n'}u(z')ds'.$$

On the other hand, replacing v by γ gives

$$u(z) = \lambda \iint \gamma(z,z')u(z')dx'dy' + \int \Delta' u(z')\frac{\partial}{\partial n'}\gamma(z,z')ds'$$

If u is a solution for the simply supported plate, *i.e.* $\Delta u = 0$ on ∂D, then

$$u = \lambda \iint \gamma u dx'dy' \qquad (3.5)$$

$$= \lambda \iint \Gamma u dx'dy' - \int \Delta' \Gamma \frac{\partial u}{\partial n'}ds'$$

$$= u_1 + u_2,$$

where $u_1(z) = \lambda \iint \Gamma(z,z')u(z')dx'dy'$ satisfies

$$\Delta^2 u_1 = \lambda u \text{ in D}$$

$$u_1 = \frac{\partial u_1}{\partial n} = 0 \text{ on } \partial D,$$

while $u_2(z) = -\int \Delta'\Gamma(z,z')\frac{\partial}{\partial n'}u(z')ds'$ satisfies

(a) $\Delta^2 u_2 = 0$ in D since $\Delta'\Gamma|_{\partial D'}$ is biharmonic in D.

(b) $u_2 = u - u_1 = 0$ on ∂D

(c) $\frac{\partial u_2}{\partial n} = \frac{\partial u}{\partial n}$ on ∂D.

Thus we can write

$$u_2 = -\int \Delta'\Gamma \frac{\partial u}{\partial n'}ds'$$

$$= -\int \Delta'\Gamma \frac{\partial u_2}{\partial n'}ds' \quad \text{from (c)}$$

$$\Delta u_2 = -\int \Delta\Delta'\Gamma \frac{\partial u_2}{\partial n'}ds'$$

$$= \int k \frac{\partial u_2}{\partial n'}ds'$$

*Primed symbols are taken in the variable z'.

where $k(z,z') = -\Delta\Delta'\Gamma(z,z')$. From Green's identity

$$\Delta u_2 = \int \frac{\partial k}{\partial n'} u_2 ds' - \iint (\Delta' k u_2 - k\Delta' u_2) dx'dy'.$$

$$\Delta u_2 = \iint k \Delta u_2 dx'dy' \quad (3.5)$$

from (b) and the fact that k is harmonic in D.

This property of k follows from the identity $\Delta\Delta' S = 0$ which is satisfied everywhere in $D \times D$, including $z = z'$ (see [3]). We shall now show that k is the reproducing kernel (see [4]) for the class of square integrable harmonic functions in D, which we denote by $H^2(D)$. Let

$$W_o = \{u = G\varphi | \varphi \in H^2(D)\}.$$

Since $H^2(D) \subset \Omega$ we have $W_o \subset W$. And since $H^2(D)$ with the inner product (.,.) is a Hilbert space (H. Weyl theorem) it follows that W_o with the inner product $<.,.>$ is also a Hilbert space. For any $\varphi \in H^2(D)$, $u = G\varphi \in W_o$ is biharmonic and we can use the identity (3.4) with v replaced by Γ to obtain

$$u = \int (u \frac{\partial}{\partial n'} \Delta' \Gamma - \frac{\partial u}{\partial n'} \Delta' \Gamma) ds'$$

$$\Delta u = \iint (k \frac{\partial u}{\partial n'} - u \frac{\partial k}{\partial n'}) ds'$$

$$= \iint (k\Delta' u - u\Delta' k) dx'dy'$$

by Green's identity. Since k is harmonic in $D \times D$ and $\Delta' u = -\varphi$ we finally obtain

$$\varphi = (k, \varphi) \text{ for every } \varphi \in H^2(D),$$

which means that $k(z,z')$ is the reproducing kernel of the Hilbert space $H^2(D)$ and may be represented by the infinite sum $\sum_{\upsilon=1}^{\infty} \psi_{\upsilon}(z)\psi_{\upsilon}(z')$, where $\{\psi_\upsilon\}$ is any orthonormal basis for $H^2(D)$.

From equation (3.5)

$$\Delta u_2 = \iint k\Delta' u_2 dx'dy'$$
$$= \iint k(\Delta' u - \Delta' u_1) dx'dy'$$
$$= \iint k\Delta' u dx'dy' - [\iint u_1 \Delta' k dx'dy'$$
$$+ \int (k \frac{\partial u_1}{\partial n'} - u_1 \frac{\partial k}{\partial n'}) ds']$$

$$= \iint k\Delta' u dx'dy'$$

from the properties of k and u_1. In view of (3.1b) and (3.3), and since $\Delta' u = 0$ on ∂D, we can write

$$\Delta' u = -\Delta G'^2 u$$

$$= -\lambda G u.$$

Therefore

$$\Delta u_2 = -\lambda \iint k G u dx'dy'.$$

Since $u_2 = 0$ on ∂D we can use (3.1b) again to obtain

$$u_2 = -G\Delta u_2$$

$$= \lambda \iint G k G u dx'dy'$$

$$= \lambda \iint K u dx'dy',$$

where $K(z,z') = \iiiint\limits_{D\times D} g(z,\zeta) k(\zeta,\zeta') g(\zeta',z') d\xi' d\eta' d\xi d\eta$

and $\zeta = \xi + i\eta$. Thus the representation (3.5) now takes the form

$$u = \lambda \iint (\Gamma + K) u dx'dy' \quad (3.6)$$

where $K = \gamma - \Gamma$ is the difference between the biharmonic Green's functions for the simply supported and the clamped plates. K itself may be regarded as the reproducing kernel for the Hilbert space W_o, since for any $u \in W_o \exists \varphi \in H^2(D)$ such that $u = G\varphi$ and

$$<k,u> = \iint \Delta' K \Delta' u dx'dy'$$

$$\Delta' K(z,z') = -\iint g(z,\zeta) k(\zeta,z') d\zeta d\eta$$

$$= -Gk(z,z')$$

$$\Delta' u = -\varphi$$

$$\Longrightarrow <K,u> = \iint G k \varphi dx'dy'$$

$$= G(k,\varphi)$$

$$= G\varphi$$

$$= u.$$

This reproducing property defines K uniquely, just as k is uniquely defined by its reproducing property in

$H^2(D)$. It is worth noting that k itself may be expressed as the difference between the harmonic Green's function g and Neumann's function N (see [5]), so that an interesting analogy may be drawn between γ, Γ, K and the operator Δ^2 on the one hand, and g,N,k and the operator Δ on the other. The problem we face in constructing a solution to the clamped rectangular plate is tied up with the problem of obtaining k, or equivalently, of constructing a basis for the harmonic functions in the rectangle. Such a basis exists but cannot be expressed explicitly in terms of known functions.

Although the component u_1 of u satisfies the boundary conditions of the clamped plate it is not quite the solution of the clamped plate because it is not an eigenfunction of Δ^2. However we shall now see that, in a sense, it is the projection of u on that solution in the geometry of W. First of all we have

$$\begin{aligned} <u_1,u_2> &= \iint \Delta u_1 \Delta u_2 dxdy \\ &= \iint u_1 \Delta^2 u_2 dxdy - \int (u_1 \frac{\partial}{\partial n}\Delta u_2 - \frac{\partial u_1}{\partial n}\Delta u_2) ds \\ &= 0 \\ &\Rightarrow u_1 \perp u_2 \text{ in W.} \end{aligned}$$

Let $v \in W$ be a solution of the clamped plate problem, *i.e.*

$$\Delta^2 v = \mu v \text{ in D and } \frac{\partial v}{\partial n} = 0 \text{ on } \partial D.$$

Then

$$\begin{aligned} <v,u_2> &= \iint \Delta v u_2 dxdy \\ &= \iint v \Delta^2 u_2 dxdy - \int (v \frac{\partial}{\partial n}\Delta u_2 - \frac{\partial v}{\partial n}\Delta u_2) ds \\ &= 0 \\ &\Rightarrow v \perp u_2 \\ &\Rightarrow <v,u> = <v,u_1> \end{aligned}$$

Conclusion

The two boundary conditions $\Delta u = 0$ and $\frac{\partial u}{\partial n} = 0$ on the solutions of equation (1.2) may be considered as limiting cases of the more general boundary conditions

$$\Delta u + \sigma \frac{\partial u}{\partial n} = 0 \quad (4.1)$$

with $\sigma = 0$ and $\sigma = \infty$ respectively. This new boundary condition, taken of course together with $u = 0$ on ∂D, has the following interpretation: Since $\Delta u = \frac{\partial^2 u}{\partial n^2}$ is proportional to the normal bending moment on the boundary [1], which has to be zero for the simply supported plate, equation (4.1) describes a situation in which the bending moment is proportional to the normal slope of the plate surface $\frac{\partial u}{\partial n}$ at the boundary. This clearly describes the situation when the plate edge is clamped by an elastic support, whose elastic coefficient may be measured by σ.

This case will be the subject of a separate study, which may reveal a little more about the connection between the two limiting cases treated in this paper.

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عـن ذبـذبـة الألـواح المـرنة

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أستاذ مساعد الرياضة بكلية الهندسة بجامعة الرياض سابقا،

يتناول هذا البحث ذبذبة لوح رقيق مرن للحالتين الشهيرتين للسند وهما : اللوح المثبت واللوح المسنود سندا بسيطا. وقد أمكن الحصول على بعض العلاقات بين مجموعتي الدالة الذاتية لكل من هاتين الحالتين.