# A GRAPHICAL DISPLAY OF FERRARI'S SOLUTION OF THE QUARTIC EQUATION

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يحتوي هذا البحث عرضاً بيانياً الحل الكلاسيكي لمعادلة الدرجة الرابعة والذي يعود الى فراري . هناك حلان اضافيان يظهران في كل حالة و لم يتطرق اليهما فراري ، وقد بحث هذان الحلان وعرضا في رسم واحد .

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*The classical solution of the quartic equation, due to Ferrari, is shown in graphical form.*

*The two alternative solutions which arise in every case, and which are discarded in the Ferrari solution, are shown in the same graphical form, and are discussed.*

## NOMENCLATURE

| | |
|---|---|
| B, C, D, E | coefficients of a quartic : real numbers |
| Z | coefficient: complex |
| a, b, c, d | coefficients of the quadratic factors of a quartic: real and/or complex. |
| N | coefficient of $x^0$ in the expression $(x^2 + B\frac{x}{2} + N)_2$ |
| $n_1, n_3, n_2,$ | the three possible values in the expression $(x^2 + B\frac{x}{2} + n)^2$ which will solve the quartic |
| $-p, -q, -r, -s$ | the roots of the quartic: real and/or complex |
| $\alpha$ | the starting point of a figure representing a polynomial. |
| $\gamma$ line | the locus of the junction points of $x^1$ and $x^0$ in the figures representing $(x^2 + \frac{B}{2}x + N)^2$ |
| $\delta$ line | the line drawn parallel to the $\gamma$ line, at a vertical distance of E units from it. |
| $\lambda, \mu$ | the coefficients of Ferrari's added expression, before squaring. |
| $\omega$ | the terminal point of a figure representing a polynomial. |
| $\omega$ curve | the locus of $\omega$ in the figures representing $(x^2 + \frac{B}{2}x + N)^2$. |

### The Graphical Notation

The notation is not part of this work, and is therefore described separately in Appendix 1. It is a self-consistent system, true for all combination of signs. For case of reading in this work only the quartic with positive signs is used.

## THE CONSTRUCTION OF A DIAGRAM AND ITS PARTS

### The representation of the quartic

The typical quartic expression consists of five straight lines at right angles to each other, beginning at $\alpha$ and ending at $\omega$ , as in Fig.1.

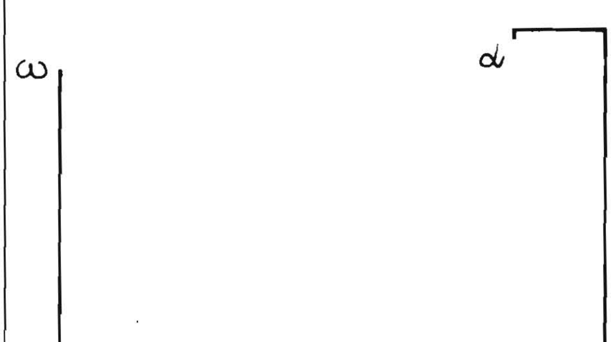


Fig. 1

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**The $\omega$ curve**

Consider the expression $(x^2 + \frac{B}{2}x + N)^2$, and plot $\omega$ for all values of N. This will produce the curve which is seen to be symmetrical about a vertical line through $\alpha$. See Fig. 2.

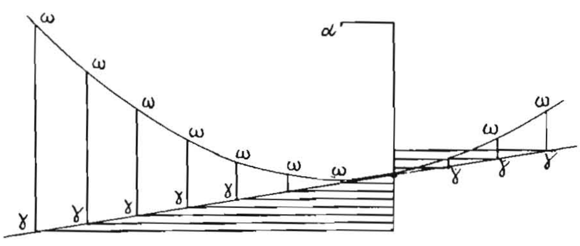


Fig. 2

If the lowest point on the curve (N=1) is taken as the origin, then

$$y = N^2 - 2N + 1$$
$$x = B(1-N)$$

and hence $y = \frac{(x)^2}{B}$ See also Appendix 3.

**The $\gamma$ line**

Consider the point $\gamma$ on each figure representing $(x^2 + \frac{Bx}{2} + N)^2$. This expression expands to

$$x^4 + Bx^3 + \left(\frac{B^2}{4} + 2N\right)x^2 + BNx + N^2$$

Then the slope between any two $\gamma$ points is

$$\frac{\left(\frac{B^2}{4} + 2N_1\right) - \left(\frac{B^2}{4} - 2N_o\right)}{BN_1 - BN_o} = \frac{2}{B}$$

and hence the points lie on a straight line, the $\gamma$ line. See Fig. 2.

**The $\delta$ line**

Now draw a line parallel to the $\gamma$ line and above it, at a vertical distance E. Let this be the $\delta$ line, the significance and purpose of which is as follows.

The bi-quadratic which is the object of the Ferrari solution will have its $\gamma$ point on the $\gamma$ line, and its final term will be a vertical straight line from the $\gamma$ line to the $\omega$ curve. The $\delta$ line, in intercepting this, shows how much of this term comes from the given quartic (i.e., the portion below the $\delta$ line) and how much be supplied by the added quadratic, which has the characteristic shape shown in Fig. 3. See also Fig. 13.

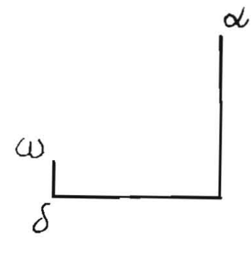


Fig. 3

It follows that the $\delta$ point of each of the three quadratics which will complete the satisfying bi-quadratic, will lie on the $\delta$ line.

**HISTORY**

If the quartic is expressed as $x^4 + Bx^3 + Cx^2 + Dx + E = 0$, then Ferrari's method is based on the completion of square thus:

$$\begin{aligned} x^4 + Bx^3 + cx^2 + Dx + E &= 0 \\ \underline{\lambda^2x^2 + 2\lambda\mu x + \mu^2} &= \underline{\lambda^2x^2 + 2\lambda\mu x + \mu^2} \\ x^4 + Bx^3 + (C+\lambda^2)x^2 & \\ + (D + 2\lambda\mu)x + (E + \mu^2) &= \lambda^2x^2 + 2\lambda\mu x + \mu^2 \\ \text{or} \quad (x^2 + \frac{Bx}{2} + N)^2 &= (\lambda x + \mu)^2 \end{aligned}$$

In the left hand expression, only N is unknown, and $N^2 = E + \mu^2$. Ferrari's method sets up a cubic equation which finds values of $\mu$. One at least of the three values is real. This is the value used by Ferrari. Let this be called the Main Solution, and let $\sqrt{E+\mu^2}$ be called $n_1$.

**The Graphical Display of the Main Solution**

In Fig. 4, the graphical figure representing the given quartic is shown.

Then to this figure is added graphically a figure which (i) represents $(\lambda^2x^2 + 2\lambda\mu x+\mu^2)$, and (ii) has its terminal point on the $\omega$ curve.

Hence we have found the required bi-quadratic, some of which is shown chain dotted.

This completes the display of the Main Solution.

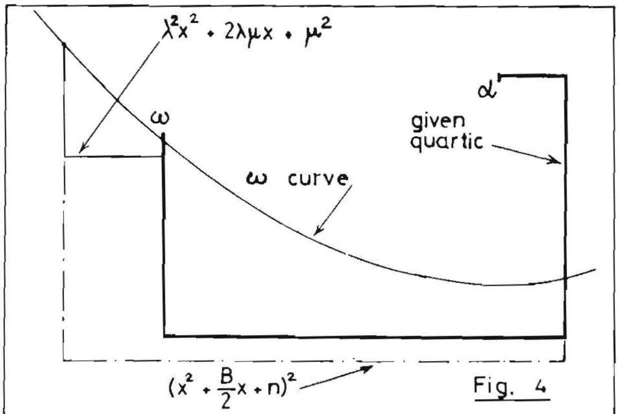

Fig. 4

**The General Solutions**

Let the original quartic have real coefficients, but not necessarily real roots. Then:

$$= x^4 + Bx^3 + Cx^2 + Dx + E$$
$$= (x + p)(x + q)(x + r)(x + s)$$
$$= (x^2 + a_1x + b_1)(x^2 + c_1x + d_1)$$
or $$= (x^2 + a_2x + b_2)(x^2 - c_2x + d_2)$$
or $$= (x^2 + a_3x + b_3)(x^2 + xc_3x + d_3)$$

Take a general pair $(x^2 + ax + b)(x^2 + cx + d)$ and make a new quadratic in which the coefficients of $x^1$ and $x^0$ are the arithmetic means of those in the general pair, i,e.

$$x^2 + \frac{a+c}{2}x + \frac{b+d}{2}$$

Then the general pair can be recovered by

$\pm\left(\frac{a-c}{2}x + \frac{b-d}{2}\right)$ and it is easy to show that:

$$(x^2 + \frac{a+c}{2}x + \frac{b+d}{2})^2 - \left(\frac{a-c}{2}x + \frac{b-d}{2}\right)^2 =$$

$(x^2 + ax + b)(x^2 + cx + d)$ or

$$(x^2 + \frac{B}{2}x + N)^2 = (x^4 + Bx^3 + Cx^2 + Dx + E) +$$
$$(\lambda x + \mu)^2$$

which is to say, the Ferrari solution.

In all cases, $\frac{a+c}{2}$ is constant, $= \frac{B}{2}$. But $\frac{b+d}{2}$ has three values, and these are the successful values $n_1$, $n_2$, and $n_3$. At least one of these values produces a new quartic such that a perfect quadratic can be ADDED to the given quartic and terminate in coincidence with the newquartic. This is the Main Solution.

The other two values of n are affected by the nature of the roots of the given quartic.

(1) If all the roots are real, then all the roots of the cubic equation are real, and all the three values of n produce a situation in which the completing quadratic is to be ADDED graphically, as shown in Fig. 5.

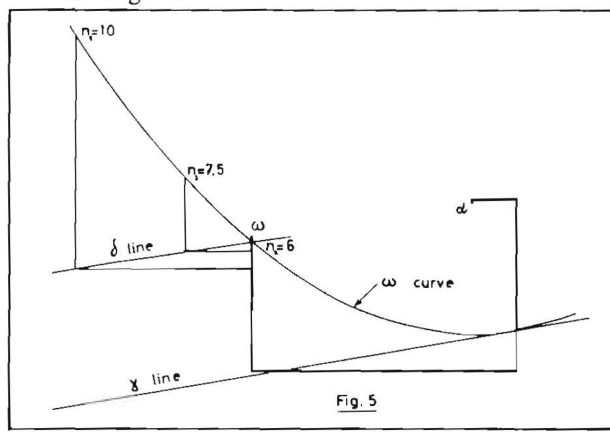

Fig. 5

$$x^4 + 12x^3 + 47x^2 + 72x + 36$$
$$=(x^4+12x^3+56x^2+120x+100)-(9x^2+48x+64) \quad \ldots n_1 = 10$$
$$=(x^4+12x^3+48x^2+72x+36)-(x^2) \quad \ldots n^2 = 6$$
$$=(x^4+12x^3+51x^2+90x+56.25)-(4x^2+18x+20.25) \quad \ldots n_3=7.5$$

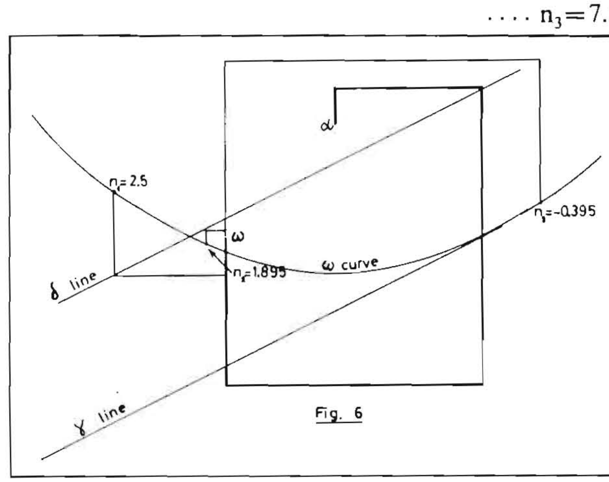

Fig. 6

$$x^4+4x^3+8x^2+7x+4$$
$$=(x^4+4x^3+9x^2+10x+6.25)-(x^2+3x+2.25) \quad \ldots n_1=2.25$$
$$=x^4+4x^3+7.79x^2+7.58x+3.59)+(0.21x^2-58x+0.41) \quad \ldots n_2 \doteq 1.895$$
$$=(x4+4x^3+3.21x^2+1.58x+0.16)+4.79x^2+8.58x +3.84 \quad \ldots n_3=-0.395$$

(2) If all the roots are complex, then one value of n will allow a quadratic to be ADDED, but the other two will produce a situation in whichthe quadratics are to be SUBTRACTED graphically, as shown in Fig. 6.

(3) If two roots are real and two complex, then one value of n will allow a quadratic to be ADDED. The other two n's will themselves be complex, and a new situation arises, as described in Appendix 3. It is confusing to show all the three solutions on the one diagram, and so they are separated into Figs. 7, 8, and 9, as explained in Appendix 3.

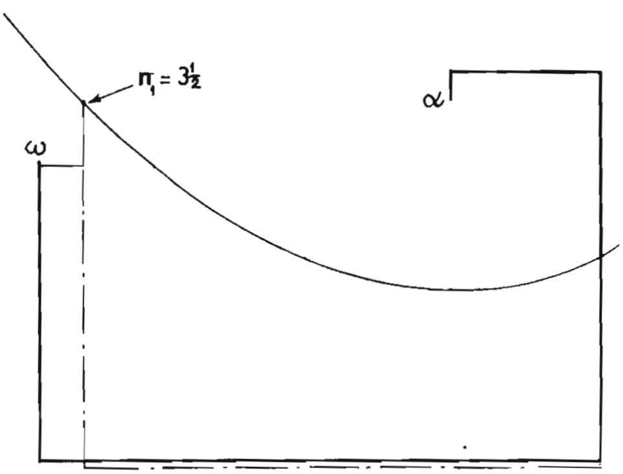


Fig. 7

$$x^4+5x^3+13x^2+19x+10$$
$$=(x^2+2\tfrac{1}{2}x+3\tfrac{1}{2})^2+(\tfrac{1}{2}x-1\tfrac{1}{2})^2$$

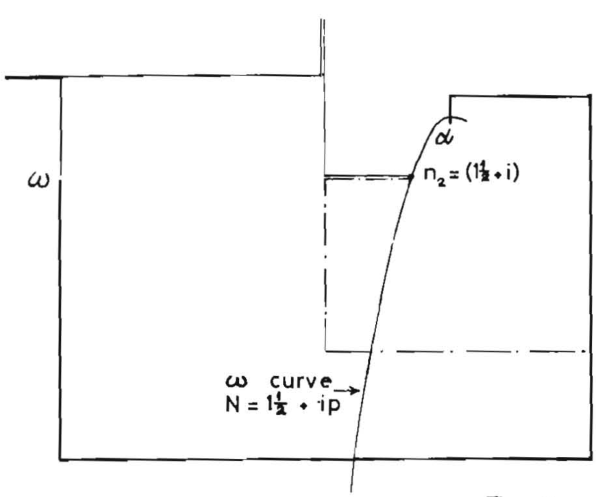


Fig. 8

$$x^4+5x^3+13x^2+19x+10$$
$$=[x^2+2\tfrac{1}{2}x+(1\tfrac{1}{2}+i)]^2-[(\tfrac{1}{2}+i2)x+(\tfrac{1}{2}+i3)]^2$$

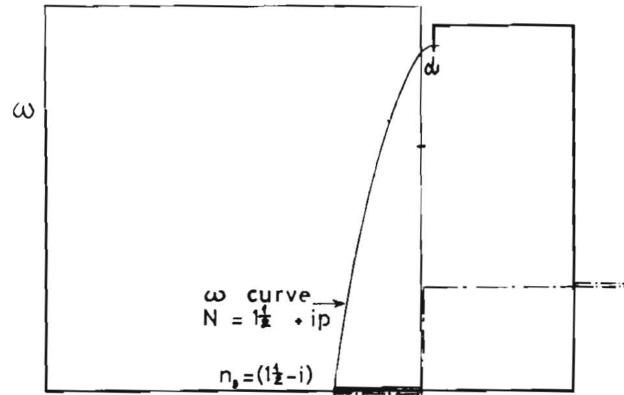


Fig. 9

$$x^4+5x^3+13x^2+19x+10$$
$$=[x^2+2\tfrac{1}{2}+(1\tfrac{1}{2}-i)]^2-[(\tfrac{1}{2}-i2)x+(\tfrac{1}{2}-i3)]^2$$

**CONCLUSION**

The graphical notation allows an interesting display of the original Ferrari solution, and links it with the other two solutions in a meaningful way.

**Appendix 1**

This does not set out to be a complete treatise on the graphical representation of polynomials and their manipulation. Here is only attempted enough (without proof) to make the main work understandable.

A polynomial can be successfully represented by a series of straight lines at right angles to each other. where the lengths of the lines are proportional to the coefficients of the terms, as shown in Fig. 7.

Addition or subtraction is performed graphically by adding or subtracting lengths of lines to or from the corresponding lines. It is convenient, if only the original and terminal points are required, to add or subtract the whole figure. See Fig. 8.

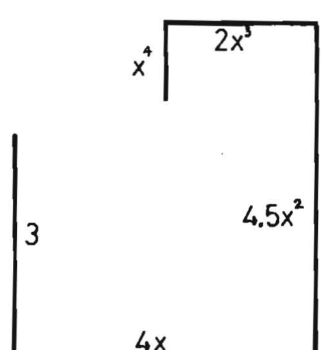


Fig. 10

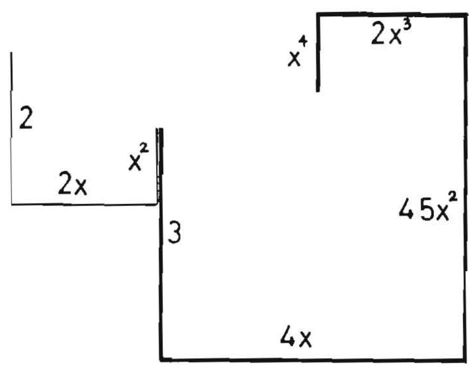


Fig. 11

Multiplication is performed by taking the **shape** of the multiplier, and adding it to each line of the original figure in a systematic manner. Then the new total figure represents the product. See Fig. 9.

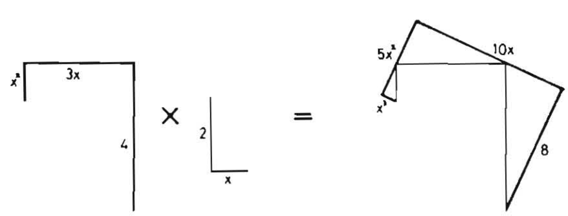


Fig. 12

It follows that when a quadratic is a perfect square, the point of the angle representing the linear term must be in the middle of the line representing x in the quadratic. See Fig. 10.

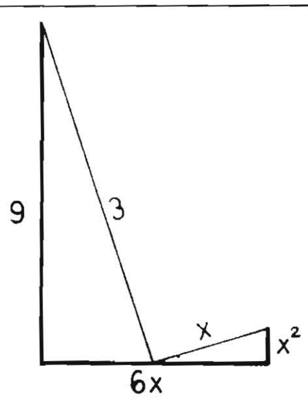


Fig. 13

**Appendix 2**

In a privately published study by the author, it is shown how the above known graphical method can be extended in a simple way to take into account imaginary parts. This can be summarised as follows.

The imaginary part of a complex number is drawn to the same scale, but placed at right angles to the real part, turning clockwise if the sign is positive, and counter clockwise if the sign is negative, In Fig. 14 we have $x^2 + (a + ib)x + (c - id)$.

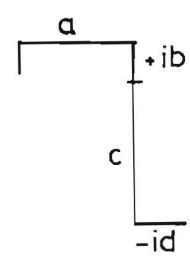


Fig. 14

**Appendix 3**

If two of the quartic roots are real and two complex then these two will be conjugate, for real coefficients of the quartic.

Only one n will be real, and will have a relationship with the $\omega$ curve as already defined. The other two values of n will be complex and conjugate, and therefore a different kind of $\omega$ curve is required, based on

$$\left(x^2 + \frac{B}{2}x + Z\right)^2$$

Such an $\omega$ curve is shown in Figs. 8 and 9, which (together with Fig. 7) are based on the numerical example:

$$x^4 + 5x^3 + 13x^2 + 19x + 10 = 0$$

which factorizes into $(x+1)(x+2)(x+1+i2)(x+1-i2)$ for which

$n_1 = 3\frac{1}{2}$, the quadratic being $(\frac{1}{2}x - 1\frac{1}{2})^2$

$n_2 = 1\frac{1}{2} + i$, the quadratic being $-[(\frac{1}{2} + i2)x + (\frac{1}{2} + i3))]^2$

$n_3 = 1\frac{1}{2} - i$, the quadratic being $-[(\frac{1}{2} - i2)x + (\frac{1}{2} - i3))]^2$