MOTION OF RIGID BODIES HAVING AN INERTIA ELLIPSOID OF REVOLUTION

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حركة الاجسام المتماسكة ذات عزم قصور ذاتي بيضاوي

يتضمن هذا البحث حلا كاملا لمعادلات أويلر الديناميكية لحركة جسم متماسك ذي عزم قصور ذاتي على شكل بيضاوي وتحت تأثير عزوم مثبتة في الجسم ومتغيرة بالنسبة للزمن وقد استخدمت هذه الطريقة في ايجاد الحل عندما تكون العزوم ثابتة في المقدار ويلاحظ حتى في هذه الحالة البسيطة ان الحل يتضمن تكامل فرزنل للجيب وجيب التمام .

Abstract

This paper presents a complete solution to the Euler dynamical equations governing the motion of a rigid body having an inertia ellipsoid of revolution and acted on by body-fixed moments which are general functions of time. As an example, the solution for the case of constant body-fixed moments is presented and it is worthwhile to note that even for this simple case the solution involves Fresnal sine and cosine integrals.

1. Introduction

The Euler dynamical equations governing the rotational motion of rigid bodies have not, as yet, been solved in general form. Many special cases have been discussed in literature. The simple case of moment-free rigid body has been solved [1], and the same procedure has been shown [2] to apply for the case when the applied couple is proportional to the angular momentum. A procedure is suggested [3,4] for solution when the applied moment is along one of the principal axes, and for the general case of a symmetrical rigid body. The purpose of this paper is to present a complete solution for the case of a rigid body having an inertia ellipsoid of revolution under the influence of body-fixed moments which are general functions of time.

2. Analysis

The Euler dynamical equations governing the motion of a rigid body having an inertia ellipsoid of revolution are

$$I\dot{\omega}_1 - (I - J)\,\omega_2\,\omega_3 = T_1 \tag{1}$$

$$I\dot{\omega}_2 - (J - I)\,\omega_3\,\omega_1 = T_2 \tag{2}$$

$$J\dot{\omega}_3 = T_3 \tag{3}$$

where I and J denote the principal moments of inertia, ω_1, ω_2, and ω_3 are the components of the angular velocity along the principal body-fixed axes, and T_1, T_2, and T_3 are the components of the external moments along the same axes.

Equations (1), (2), and (3) can be rewritten as

$$\dot{\omega}_1 + k\,\omega_2\,\omega_3 = M_1 \tag{4}$$

$$\dot{\omega}_2 - k\,\omega_3\,\omega_1 = M_2 \tag{5}$$

$$\dot{\omega}_3 = M_3 \tag{6}$$

where M_1, M_2, M_3, and k are defined as

$$M_1 = \frac{T_1}{I} \tag{7}$$

$$M_2 = \frac{T_2}{I} \tag{8}$$

$$M_3 = \frac{T_3}{J} \tag{9}$$

$$k = \frac{J - I}{I} \tag{10}$$

The solution of Eq. (6) is

$$\omega_3 = \omega_{30} + \int_0^t M_3(\eta)\,d\eta$$

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Trnasforming the dependent and independent variables as follows:

$$\omega_1 = u_1 + \int_0^t M_1(\eta)\, d\eta \qquad (11)$$

$$\omega_2 = u_2 + \int_0^t M_2(\eta)\, d\eta \qquad (12)$$

and

$$\tau = \int_0^t \omega_3(\eta)\, d\eta \qquad (13)$$

Equations (4) and (5) are transformed into

$$u_1' + k\, u_2 = -k\, G_2 \qquad (14)$$

$$u_2' - k\, u_1 = k\, G_1 \qquad (15)$$

where

$$G_1(\tau) = \int_0^{t(\tau)} M_1(\eta)\, d\eta \qquad (16)$$

$$G_2(\tau) = \int_0^{t(\tau)} M_2(\eta)\, d\eta \qquad (17)$$

$$(\ )' = \frac{d}{d\tau}(\ ) \qquad (18)$$

Differentiating Eq. (14), multiplying Eq. (15) by (—k) and adding them, one arrives at

$$\ddot{u}_1 + k^2 u_1 = -k\, G_2' - k^2\, G_1$$

whose solution is

$$u_1 = A \sin k\tau + B \cos k\tau$$

$$- \int_0^\tau [k G_1(\eta) + G_2(\eta)] \sin k(\tau - \eta)\, d\eta \qquad (19)$$

Observing that

$$\int_0^\tau G_2(\eta) \sin k(\tau - \eta)\, d\eta$$

$$= [G_2(\eta) \sin k(\tau - \eta)]_0^\tau$$

$$+ k \int_0^\tau G_2(\eta) \cos k(\tau - \eta)\, d\eta$$

$$= k \int_0^\tau G_2(\eta) \cos k(\tau - \eta)\, d\eta \qquad (21)$$

one finds that

$$u_1 = A \sin k\tau + B \cos k\tau$$

$$- k \int_0^\tau G_1(\eta) \sin k(\tau - \eta)\, d\eta$$

$$- k \int_0^\tau G_2(\eta) \cos k(\tau - \eta)\, d\eta \qquad (22)$$

Substituting from Eq. (22) into Eq. (14), one gets

$$u_2 = -A \cos k\tau + B \sin k\tau$$

$$+ k \int_0^\tau G_1(\eta) \cos k(\tau - \eta)\, d\eta$$

$$- k \int_0^\tau G_2(\eta) \sin k(\tau - \eta)\, d\eta \qquad (23)$$

The constants A and B can be evaluated by observing that from Equations (11) and (12)

$$\omega_{10} = u_{10} \qquad (24)$$

$$\omega_{20} = u_{20} \qquad (25)$$

from which it follows from Equations (22) and (23) that

$$A = -\omega_{20} \qquad (26)$$

$$B = \omega_{10} \qquad (27)$$

Therefore the final expressions for ω_1, ω_2, and ω_3 are

$$\omega_1 = -\omega_{20} \sin k\tau + \omega_{10} \cos k\tau$$

$$+ \int_0^t M_1(\eta)\, d\eta$$

$$- k \int_0^\tau G_1(\eta) \sin k(\pi - \eta)\, d\eta$$

$$- k \int_0^\tau G_2(\eta) \cos k(\tau - \eta)\, d\eta \qquad (28)$$

$$\omega_2 = \omega_{20} \cos k\tau + \omega_{10} \sin k\tau$$

$$+ \int_0^t M_2(\eta)\, d\eta$$

$$+ k \int_0^\tau G_1(\eta) \cos k(\tau - \eta)\, d\eta$$

$$- k \int_0^\tau G_2(\eta) \sin k(\tau - \eta)\, d\eta \qquad (29)$$

and

$$\omega_3 = \omega_{30} + \int_0^t M_3(\eta)\, d\eta \qquad (30)$$

where τ, as defined by Eq. (13), is given by

$$\tau = \omega_{30} t + \int_0^t \int_0^\eta M_3(\varepsilon)\, d\varepsilon \; d\eta \qquad (31)$$

3. Example

For the special case when the external moments consist of constant body-fixed moments only, then T_1, T_2, and T_3 are all constants. Therefore

$$\tau = \omega_{30}\, t + \frac{M_3}{2} t^2 \qquad (32)$$

Solving Eq. (32) for t, one gets

$$t = b(\sqrt{\tau + a^2} - a) \qquad (33)$$

where

$$b = \sqrt{\frac{2}{M_3}} \qquad (34)$$

$$a = \frac{\omega_{30}}{\sqrt{2 M_3}} \qquad (35)$$

The functions $G_1(\tau)$ and $G_2(\tau)$ are then

$$G_1(\tau) = M_1 t = M_1 b \; (\sqrt{\tau + a^2} - a) \qquad (36)$$

$$G_2(\tau) = M_2 t = M_2 b \; (\sqrt{\tau + a^2} - a) \qquad (37)$$

Therefore

$$\omega_1 = -\omega_{20} \sin k\tau + \omega_{10} \cos k\tau + M_1 t$$

$$- k \int_0^\tau M_1 \; b\,(\sqrt{\eta + a^2} - a) \sin k\,(\pi - \eta)\; d\eta$$

$$- k \int_0^\tau M_2 \; b\,(\sqrt{\eta + a^2} - a) \cos k\,(\tau - \eta)\; d\eta$$

$$= -\omega_{20} \sin k\,\tau + \omega_{10} \cos k\tau + M_1 t$$

$$+ a\, b\, M_1 (1 - \cos k\,\tau) + a\, b\, M_2 \sin k\,\tau$$

$$- kb\,(M_1 \sin k\pi + M_2 \cos k\tau) \int_0^\tau \sqrt{\eta + a^2} \cos k\,\eta \; d\eta$$

$$+ kb\,(M_1 \cos k\tau - M_2 \sin k\tau) \int_0^\tau \sqrt{\eta + a^2} \sin k\,\eta\, d\eta \qquad (38)$$

$$\omega_2 = \omega_{20} \cos k \; \pi + \omega_{10} \sin k\pi + M_2 t$$

$$+ k \int_0^\tau M_1 b\,(\sqrt{+a^2} - a) \cos k\,(\tau - \eta)\; d\eta$$

$$- k \int_0^\tau M_2 b\,(\sqrt{\eta + a^2} - a) \sin k\,(\tau - \eta)\; d\eta$$

$$= \omega_{20} \cos k\pi + \omega_{10} \sin k\pi + M_2 t$$

$$- a\, b\, M_1 \sin k\tau + a\, b\, M_2 (1 - \cos k\,\tau)$$

$$+ kb\,(M_1 \cos k\pi - M_2 \sin k\tau) \int_0^\tau \sqrt{\eta + a^2} \cos k\,\eta \; d\eta$$

$$+ kb\,(M_1 \sin k\tau + M_2 \cos k\tau) \int_0^\tau \sqrt{\eta + a^2} \sin k\,\eta\, d\eta \qquad (39)$$

and

$$\omega_3 = \omega_{30} + M_3 t \qquad (40)$$

The integrals appearing in equations (38) and (39) will now be evaluated.

Let

$$I_c = \int_0^\tau \sqrt{\eta + a^2} \cos k\,\eta \; d\eta \qquad (41)$$

and

$$I_s = \int_0^\tau \sqrt{\eta + a^2} \sin k\,\eta \; d\eta \qquad (42)$$

Then

$$I_c = \left[\frac{1}{k} \sqrt{\eta + a^2} \; \sin k\,\eta \;\right]_0^\tau - \frac{1}{2k} \int_0^\tau \frac{\sin k\,\eta}{\sqrt{\eta + a^2}}\, d\eta$$

$$= \frac{1}{k} \sqrt{\tau + a^2} \sin k\,\tau$$

$$- \frac{1}{2k} \int_0^\tau \frac{\sin k\eta}{\sqrt{\eta + a^2}}\, d\eta \qquad (43)$$

Using

$$Y = \eta + a^2$$

one gets

$$I_c = \frac{1}{k} \sqrt{\tau + a^2} \; \sin k\tau \; -$$

$$\frac{1}{2k}\int_{a^2}^{\tau+a^2}\frac{1}{\sqrt{Y}}\sin k\,(y-a^2)\,dy$$

$$=\frac{1}{k}\sqrt{\tau+a^2}\sin k\tau -$$

$$\frac{1}{2k}\int_{a^2}^{\tau+a^2}\frac{1}{\sqrt{Y}}(\cos ka^2 \sin ky-\sin ka^2\cos ky)dy$$

$$=\frac{1}{k}\sqrt{\tau+a^2}\sin k\tau$$

$$-\frac{1}{2k}\cos ka^2\int_{a^2}^{\tau+a^2}\frac{1}{\sqrt{y}}\sin ky\,dy$$

$$+\frac{1}{2k}\sin ka^2\int_{a^2}^{\tau+a^2}\frac{1}{\sqrt{Y}}\cos ky\,dy$$

$$=\frac{1}{k}\sqrt{\tau+a^2}\sin k\tau$$

$$-\frac{1}{2k}\sqrt{\frac{2\pi}{k}}\left[S\left(\sqrt{\frac{2k}{\pi}(\tau+a^2)}\right)\right.$$

$$\left.-S\left(\sqrt{\frac{2k}{\pi}}a\right)\right]\cos ka^2$$

$$+\frac{1}{2k}\sqrt{\frac{2\pi}{k}}\left[C\left(\sqrt{\frac{2k}{\pi}(\tau+a^2)}\right)\right.$$

$$\left.-C\left(\sqrt{\frac{2k}{\pi}}a\right)\right]\sin ka^2 \qquad (44)$$

where S(x) and C(x) are the Fresnal sine and cosine integrals which are defined as

$$S(x)=\int_0^x \sin\frac{\pi t^2}{2}dt \qquad (45)$$

and

$$C(x)=\int_0^x \cos\frac{\pi t^2}{2}dt \qquad (46)$$

Also

$$I_s=\left[-\frac{1}{k}\sqrt{\eta+a^2}\cos k\eta\right]_0^{\tau}$$

$$+\frac{1}{2k}\int_0^{\tau}\frac{\cos k\eta}{\sqrt{\eta+a^2}}d\eta$$

$$=\frac{a}{k}-\frac{1}{k}\sqrt{\tau+a^2}\cos k\tau$$

$$+\frac{1}{2k}\int_{a^2}^{\tau+a^2}\frac{1}{\sqrt{Y}}\cos k(y-a^2)\,dy$$

$$=\frac{a}{k}-\frac{1}{k}\sqrt{\tau+a^2}\cos k\tau$$

$$+\frac{1}{2k}\sqrt{\frac{2\pi}{k}}\left[S\left(\sqrt{\frac{2k}{\pi}(\tau+a^2)}\right)\right.$$

$$\left.-S\left(\sqrt{\frac{2k}{\pi}}a\right)\right]\sin ka^2$$

$$+\frac{1}{2k}\sqrt{\frac{2\pi}{k}}\left[C\left(\sqrt{\frac{2k}{\pi}(\tau+a^2)}\right)\right.$$

$$\left.-C\left(\sqrt{\frac{2k}{\pi}}a\right)\right]\cos ka^2 \qquad (47)$$

Therefore the final expression for ω_1 and ω_2 are

$$\omega_1=-\omega_{20}\sin k\left(\omega_{30}t+\frac{M_3}{2}t^2\right)$$

$$+\omega_{10}\cos k\left(\omega_{30}t+\frac{M_3}{2}t^2\right)$$

$$+\sqrt{\frac{\pi}{2k}}bM_1\left\{S\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left.-S\left[\sqrt{\frac{2k}{\pi}}a\right]\right\}\sin k\left(\frac{t}{b}+a\right)^2$$

$$+\sqrt{\frac{\pi}{2k}}bM_1\left\{C\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left.-C\left[\sqrt{\frac{2k}{\pi}}a\right]\right\}\cos k\left(\frac{t}{b}+a\right)^2$$

$$+\sqrt{\frac{\pi}{2k}}bM_2\left\{S\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left.-C\left[\sqrt{\frac{2k}{\pi}}a\right]\right\}\cos k\left(\frac{t}{b}+a\right)^2$$

$$-\sqrt{\frac{\pi}{2k}}bM_2\left\{C\left[\sqrt{\frac{2k}{\pi}}\frac{t}{b}+a\right)\right.$$

$$\left.-C\left[\sqrt{\frac{2k}{\pi}}a\right]\right\}\sin k\left(\frac{t}{b}+a\right)^2 \qquad (48)$$

and

$$\omega_2=\omega_{20}\cos k\left(\omega_{30}t+\frac{M_3}{2}t^2\right)$$

$$+\omega_{10}\sin k\left(\omega_{30}t+\frac{M_3}{2}t^2\right)$$

$$-\sqrt{\frac{\pi}{2k}}\, bM_1 \left\{ S\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left. - S\left[\sqrt{\frac{2k}{\pi}}\, a\right]\right\} \cdot \cos k\left(\frac{t}{b}+a\right)^2$$

$$+\sqrt{\frac{\pi}{2k}}\, bM_1 \left\{ C\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left. - C\left[\sqrt{\frac{2k}{\pi}}\, a\right]\right\} \sin k\left(\frac{t}{b}+a\right)^2$$

$$+\sqrt{\frac{\pi}{2k}}\, bM_2 \left\{ S\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left. - S\left[\sqrt{\frac{2k}{\pi}}\, a\right]\right\} \sin k\left(\frac{t}{b}+a\right)^2$$

$$+\sqrt{\frac{\pi}{2k}}\, bM_2 \left\{ C\left[\sqrt{\frac{2k}{\pi}}\left(\frac{t}{b}+a\right)\right]\right.$$

$$\left. - C\left[\sqrt{\frac{2k}{\pi}}\, a\right]\right\} \cos k\left(\frac{t}{b}+a\right)^2 \qquad (49)$$

References

1. *Synge, J.L. and Griffith, B.A., Principles of Mechanics McGraw-Hill, New York, 1959, pp. 377-380.*

2. *Greenhill, A.G., The Applications of Elliptic Functions, Dover Publications, New York, 1959, p. 106.*

3. *Dasarathy, B.V., "A Nonlinear Problem in Rigid Body Dynamics," AIAA, Vol. 8, No. 10, Oct. 1970, pp. 1870-1871.*

4. *Magnus, K., Gyrodynamics, International Centre for Mechanical Sciences, Udine, 1974, pp. 28-30.*