USE OF RUTISHAUSER METHOD FOR ENTRANCE REGION HEAT TRANSFER IN PLANE COUETTE FLOW OF A POWER LAW FLUID

A.S. El-Ariny*
A. Aziz*

استخدام طريقة روتشاهاوزر لانتقال الحرارة عند المدخل لسريان كوايت مستوى لمائع يتبع قانون أسي

استخدمت في هذا البحث طريقة عددية تتضمن طريقة القيمة الخاصة لروتشاهاوزر لحل معادلة انتقال الحرارة عند منطقة المدخل لحالة سريان كوايت مستوى لمائع لا يتبع قانون نيوتن . وقد فرض انه عند المدخل يناظر شكل توزيع السرعة معدل تغير الضغط للوضع النهائي للسريان .

اما شكل توزيع درجة الحرارة عند المدخل فقد فرض أنه ناتج عن التشتت اللزج الذي يتم في مجرى سابق بين مستويين متوازيين ، ذوي درجة حرارة محددة هـ صفر . وعند دخول المائع منطقة المدخل الحراري يحدث له تغير مفاجيء في درجة الحرارة من هـ صفر الى هـ١ عند المستوى المتحرك العلوي في حين يبقى المستوى السفلي الثابت عند نفس درجة الحرارة . وقد حلت معادلة الطاقة التي تحوي حدا للتشتت اللزج ، عددياً ، وذلك لامكان استنتاج التطور الحراري للانسياب مسبقاً . وقد اعطيت النتائج الخاصة بتطور درجة الحرارة في الاتجاهات الثلاثة ، ودرجة حرارة المائع ككل وكذلك عدد ناسلت للمستوى العلوي ، وذلك لقيم محددة لعوامل تدرج الضغط والتشتت اللزج ، وهذه النتائج معطاة لاسس مختلفة هي ن = ٠,٥ (شبه لدن)، ن = ٢ (سريع الامتداد)، ن = ١ (نيوتوني) .

وقد نوقشت هذه النتائج على ضوء مزايا طريقة الحساب وامتداداتها .

Abstract

A numerical scheme which incorporates the Rutishauser eigenvalue method is used to solve for thermal entrance region heat transfer in plane Couette flow of a power law non-Newtonian fluid. At entrance the prescribed velocity profile corresponds to pressure gradient assisted, fully developed Couette flow situation. The temperature profile at entrance is assumed to arise as a result of viscous dissipation in a preceding parallel plate channel with both plates at fixed temperature T_0. *As the fluid enters the thermal entrance region, it is allowed to experience a sudden change in temperature from* T_0 *to* T_1 *at the upper moving plate while the lower stationary plate is maintained at* T_0. *The energy equation which includes viscous dissipation term is solved numerically to predict the thermal development of the flow. For fixed values of pressure gradient and viscous dissipation parameters, results for developing spatial temperature, cupmixing temperature and upper plate Nusselt number are given for power law exponent* $n = 0.5$ *(pseudoplastic),* $n = 2$ *(dilatant) and* $n = 1$ *(Newtonian). These results are discussed together with the merits of the computational procedure and its extensions.*

Nomenclature

a	distance between plates
A	coefficient matrix, equation (18)
A^t	transpose of A
c	specific heat

* *Associate Professor, Mechanical Engineering Department, College of Engineering, University of Riyadh, Saudi Arabia.*

C_j	jth expansion coefficient, equations (17,19)
Ec	Eckert number = $V^2/c\,(T_1 - T_0)$
$g_j(y)$	jth eigenfunction, equation (17)
g_{ij}	ith element of jth eigenvector
h	heat transfer coefficient
k	thermal conductivity
n	exponent of power law model, equations (3,4)
Nu	Nusselt number = ha/k
$\bar{p}$	pressure
p	non-dimensional pressure = $\bar{p}/\rho V^2$
P	non-dimensional pressure gradient parameter = $-\frac{1}{Pe}\frac{dp}{dx}$
Pe	Peclet number = $V\,a/\alpha$
Pr	Prandtl number = $\mu c/k$
Re	Reynolds number = $\rho V\,a/\sigma$
s	eigenvectors associated with A^t
S_{ij}	ith element of jth eigenvector
$\bar{T}$	temperature
T	$(\bar{T} - T_0)/(T_1 - T_0)$
T_0	lower plate temperature
T_1	upper plate temperature
$\bar{u}$	velocity
u	non-dimensional velocity = $\bar{u}/V$
V	velocity of upper plate
$\bar{x}$	streamwise coordinate
x	non-dimensional streamwise coordinate = $\bar{x}/aPe$
$\bar{y}$	transverse coordinate
y	non-dimensional transverse coordinate = $\bar{y}/a$
α	thermal diffusivity = $k/\rho c$
σ_0	power law parameter, equation (3)
σ	$\sigma_0(\frac{V}{a})^{n-1}$
ϕ,θ,ψ	non-dimensional temperatures, equations (5c,9)
$\bar{\tau}$	shear stress
τ	non-dimensional shear stress = $\bar{\tau}/\rho V^2$
λ_j	jth eigenvalue
ρ	density

Subscripts

i	general grid point
m	cup-mixing

Superscripts

t	transpose

Introduction

The analysis of entrance region heat transfer in plane Couette flow of a Newtonian fluid has been presented by Sestak and Rieger [1] and Bruin [2] among others. Despite the assumption of fully developed velocity profile at entrance, the analytical solution of the energy equation entails computational difficulties particularly if the effect of additional pressure gradient. is included [2]. To alleviate the difficulties associated with the computation of the eigenvalues and subsequently the expansion coefficients, a numerical scheme incorporating the Rutishauser matrix transformations was developed by the present authors [3]. The merits of this numerical procedure are computational simplicity, accuracy and its capability to handle different types of boundary conditions. The original work [3] has been recently extended to include non-Newtonian behaviour [4] for which the mathematical form of the velocity profile makes the analytical approach extremely complicated if not impossible. In [4] a two-parameter Prandtl-Eyring model was employed to describe the pseudoplastic behaviour. The present work extends the study to a power law model which permits the simulation of both psuedo-plastic and dilitant behaviours. The results obtained are therefore of wider applicability. Furthermore the present analysis improves upon the earlier one [4] by allowing for viscous dissipation upstream in establishing the temperature profile at the entrance.

Sample results for spatial temperature, cup-mixing temperature and upper plate Nusselt number are presented for fixed values of pressure gradient and viscous dissipation parameters. These results demonstrate the influence of power law exponent n on thermal development of flow. The values of n chosen are 0.5 (psuedoplastic), 2.0 (dilatant) and 1.0 (Newtonian) to compare the results for three different types of fluid behaviour. These results are discussed together with the possible extensions of the procedure to include temperature dependent viscous properties or axial heat conduction effect.

Problem description

We consider steady Couette flow of an incompressible viscous non-Newtonian fluid whose behaviour is described by a power law model [5]. As depicted in the insert (Fig. 1) the plates are separated by a distance a. The lower plate is stationary while the upper plate moves with a uniform velocity V. The upper and lower plates are maintained at constant temperatures T_1 and T_0 respectively. At entrance ($x = 0$) the velocity profile is assumed to be fully developed corresponding to pressure gradient assisted flow situation. This assumption is valid for most non-Newtonian fluids because of high Prandtl numbers [6]. Since viscous dissipation is allowed for in the thermal entrance region, a similar allowance is made upstream ($x<0$) where the flow develops hydrodynamically. We assume that the entrance region is preceded by a parallel plate channel with both plates kept at constant temperature T_0. As result of viscous dissipation in this channel, the fluid emerges with a fully developed temperature profile and enters the thermal entrance region. With both velocity and temperature profiles thus prescribed at $x=0$, the fluid undergoes a sudden change in temperature from T_0 to T_1 at the upper moving plate, the lower stationary plate being maintained at T_0. The analysis aims to predict the subsequent development of the temperature profile. The fact that only the specific problem just described is solved here does not reflect the limitation of the numerical procedure presented in the next section. In fact, as discussed later, the method is applicable to a wide class of thermal entrance problems.

Formulation

The pertinent momentum and energy equations for the foregoing problem can be written in dimensionless form as

$$\frac{d\tau}{dy} = -\frac{1}{Pe}\frac{dp}{dx} \qquad (1)$$

$$u\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} + EcPe\ \tau\frac{du}{dy} \qquad (2)$$

In writing the momentum equation body force and inertia terms have been neglected and the flow is assumed to be assisted by a constant pressure gradient. The form of the energy equation implies constant fluid properties and negligible axial heat conduction.

For a power law non-Newtonian fluid the shear stress and velocity gradient are related according to

$$\bar{\tau} = \sigma_0 \left(\frac{d\bar{u}}{d\bar{y}} \right)^n \qquad (3)$$

with dimensionless version as

$$\tau = \frac{1}{Re} \left(\frac{du}{dy} \right)^n \qquad (4)$$

Equations (1,2,4) are to be solved subject to the following conditions

$$y = 0,\ x>0,\ u = 0,\ T = 0 \qquad (5a)$$
$$y = 1,\ x>0,\ u = 1,\ T = 1 \qquad (5b)$$
$$x = 0,\ 0 \leqslant y \leqslant 1,\ T = T(0,y) = \phi(y) \qquad (5c)$$

All symbols appearing in the above formulation are defined in the Nomenclature and need not be repeated here.

Method of solution

Integrating the momentum equation (1) gives

$$\tau = Py + C_0 \qquad (6)$$

Further substituting equation (6) into (4) and integrating using the appropriate boundary conditions on u from equations (5a, 5b) gives the velocity profile as

$$u = \frac{n}{(n+1)P} \left[(Py + C_0)^{\frac{n+1}{n}} - C_0^{\frac{n+1}{n}} \right] \qquad (7)$$

where C_0 is given by

$$\frac{n}{(n+1)P} \left[(P+C_0)^{\frac{n+1}{n}} - C_0^{\frac{n+1}{n}} \right] - 1 = 0 \qquad (8)$$

For assigned values of n and P equation (8) was solved by bisection iterative procedure [7] to obtain C_0.

To facilitate the solution of equation (2) we introduce the transformation [8]

$$\theta(x, y) = T(x, y) - \psi(y) \qquad (9)$$

into equation (2), make use of equations (4,7) and carry out some manipulation to obtain the energy equation in the form

$$u \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2} + \frac{d^2 \psi}{dy^2} + EcPr\,(Py + C_0)^{\frac{n+1}{n}} \qquad (10)$$

The function ψ is chosen such that it satisfies

$$\frac{d^2 \psi}{dy^2} + EcPr(Py + C_0)^{\frac{n+1}{n}} = 0 \qquad (11)$$

thus simplifying equation (10) to

$$u \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2} \qquad (12)$$

In accordance with equation (9), the temperature boundary conditions (5a,5b,5c) take the form

$$y = 0,\ x>0,\ \theta = 0,\ \psi = 0 \qquad (13a)$$
$$y = 1,\ x>0,\ \theta = 0,\ \psi = 1 \qquad (13b)$$
$$x = 0,\ 0 \leqslant y \leqslant 1,\ \theta = \phi - \psi \qquad (13c)$$

The function $\phi(y)$ in equation (13c) can be specified arbitrarily. To obtain specific results, we take the profile to be that which would occur under fully developed condition as a result of viscous dissipation in a parallel plate channel with both walls maintained at T_0 (see inset in Fig. 1). This can be obtained by setting $\frac{\partial T}{\partial x} = 0$ in equation (2) and solving it for constant temperature conditions T_0 at the plates.

The solution for ψ is straightforward and is given by

$$\psi = -\frac{n^2 EcPr}{(2n+1)(3n+1)P^2} (Py+C_0)^{\frac{3n+1}{n}} + \frac{C_1}{P}(Py+C_0) + C_2 \qquad (14)$$

where

$$C_1 = 1 + \frac{n^2 EcPr}{(2n+1)(3n+1)P^2} \Big[(P +$$

$$\left. C_0)^{\frac{3n+1}{n}} - C_0^{\frac{3n+1}{n}} \right] \quad (15)$$

and

$$C_2 = \frac{n^2 EcPr}{(2n+1)(3n+1)P^2}(C_0)^{\frac{3n+1}{n}} - \frac{C_1}{P} C_0 \quad (16)$$

Following the method detailed out in [3,4] the solution of equation (12) is expressed as $\theta = f(x)g(y)$ leading to the series solution

$$\theta = \sum_j C_j \, g_j(y) \exp(-\lambda_j \, x) \quad (17)$$

where C_j are the expansion coefficients and λ_j and $g_j(y)$ are the eigenvalues and eigenvectors of the matrix

$$Ag = \lambda g \quad (18)$$

which is obtained by expressing the equation for g namely $g'' + \lambda ug = 0$ in finite difference form with suitable grid size. In (18) A denotes the coefficient matrix. For any grid point i, equation (17) becomes

$$\theta_i = \sum_j C_j \, g_{ij} \exp(-\lambda_j x) \quad (19)$$

where g_{ij} is the ith element of the jth eigenvector. The eigenvalues λ_j are obtained by applying to equation (18) the lower and upper matrices transformations based on Rutishauser method [3,4]. The gaussian elimination is next used to determine the corresponding eigenvectors g_j. To evaluate C_j it is necessary to determine the eigenvectors of transpose of A, i.e. A^t and then utilise the orthogonal property of eigenvectors of A and A^t together with the initial condition, equation(13c) which is x = 0, $\theta_i = \phi_i - \psi_i$ to give C_j as [3,4]

$$C_j = \left[\sum_{i=1}^{n} s_{ij} (\phi_i - \psi_i) \right] / \sum_{i=1}^{n} s_{ij} \, g_{ij} \quad (20)$$

where s_{ij} represents the ith element of jth eigenvector of A^t.

The subsequent evaluation of cup-mixing temperature and upper plate Nusselt number follow exactly the procedure outlined in [3,4]. The numerical computations were all carried out on HP2100S digital computer of the College of Engineering, University of Riyadh.

Results and discussion

For brevity of presentation the additional pressure gradient parameter P and viscous dissipation parameter EcPr are fixed at unity in all results. Fig. 1 shows a typical set of developing temperature profile for n = 0.5 (psuedoplastic), n= 1.0 (Newtonian) and n = 2.0 (dilatant). In early stages (x = 0.02) the profile develops faster for psuedoplastic fluid compared to dilitant fluid. However, as x increases the speed of development for dilitant fluid is such that it overtakes the psuedoplastic fluid. This can be observed from the profiles at x = 0.1. At x = ∞, the thermal development of flow is complete and the profiles conform to equation (14).

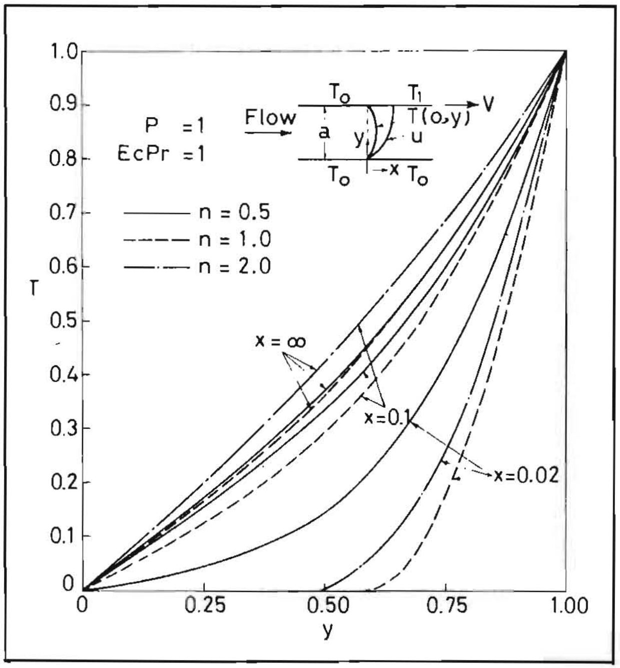

FIG. 1 Effect of exponent n on spatial temperature development.

The development of cup-mixing temperature is shown in Fig. 2. At low values of x, the value for dilatant fluid is higher than that for psuedoplastic fluid but the trend reverses at higher values of x. With the attainment of fully developed condition, the highest temperature is achieved by psuedoplastic fluid followed by dilitant and Newtonian fluids. For Newtonian fluid the cup-mixing temperature is lowest throughout the entrance region.

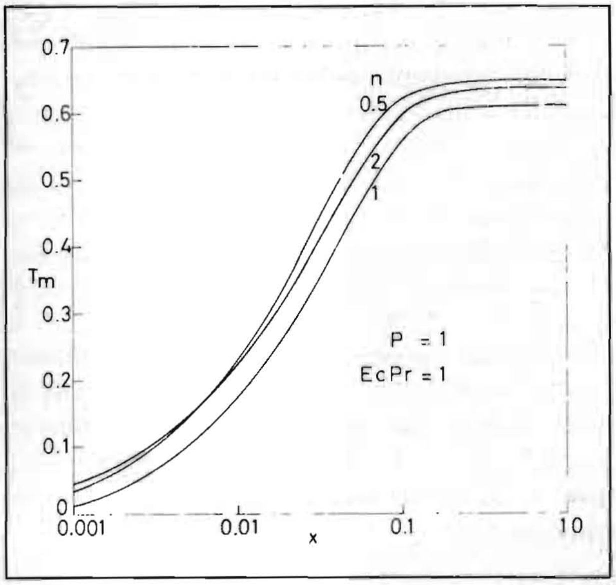

FIG. 2 Effect of exponent n on cup-mixing temperature development.

Fig. 3 contains the results for Nusselt number at the upper plate Compared to Newtonian fluid, the Nusselt number is higher for psuedoplastic fluid and lower for dilatant fluid. The same trend has been found for

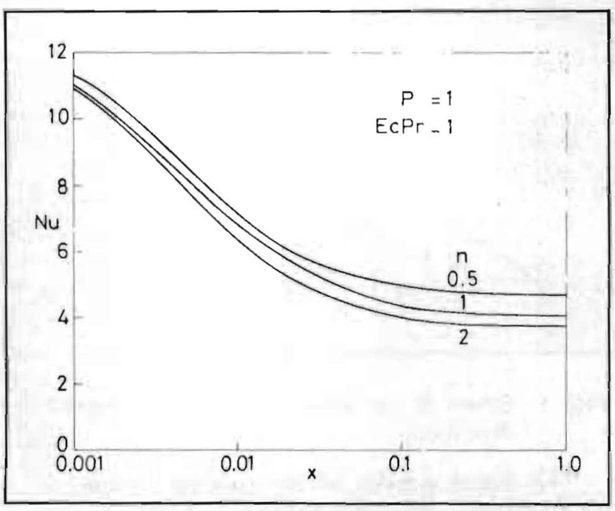

FIG. 3 Effect of exponent n on the development of Nusselt number at upper plate.

entrance region heat transfer analysis in circular tubes [9].

Concluding remarks

Although a specific problem has been solved here the computational procedure used is applicable to a wide class of thermal entrance problems. For example, it can handle flux type boundary conditions as demonstrated in [3]. It has also been used for thermal entrance region heat transfer in a circular tube for both constant temperature and constant heat flux boundary conditions [10]. The chief merits of the method are that the velocity and temperature profiles at entrance can be prescribed arbitrarily and that it gives accurate results even for a moderate size grid.

In the work presented so far we have not included the effects of axial conduction (important at low Peclet numbers) and temperature dependent viscous properties in the analysis. It is known that both these effects can significantly affect the heat ransfer rates. The authors are currently engaged with further development of the computational procedure to accomodate the aforementioned effects.

References

1. *Sestak, J. and Rieger, F., "Laminar Heat Transfer to Steady Couette Flow between Parallel Plates", International Journal of Heat and Mass Transfer, Vol. 12, 1969, pp. 71-80.*

2. *Bruin, S., "Temperature Distributions in Couette Flow with and without Additional Pressure Gradient", International Journal of Heat and Mass Transfer, Vol. 15, 1972, pp. 341-349.*

3. *El-Ariny, A.S., and Aziz A, "A Numerical Solution of Entrance Region Heat Transfer in Plane Couette Flow", Trans. ASME, Journal of Heat Transfer, Vol. 98, Series C. No. 3, 1976, pp. 427-431.*

4. *El-Ariny, A.S. and Aziz, A., "Thermal*

Entrance Region Heat Transfer in Plane Couette Flow of a Non-Newtonian Fluid". Presented at 97th ASME Winter Annual Meeting, New York, Paper No. 76-WA/HT-41.

5. *Bird, R.B., Stewart, W.E. and Lightfoot, E.N., "Transport Phenomena", John Wiley, N.Y., 1960.*

6. *Kays, W.M., "Convective Heat and Mass Transfer", McGraw-Hill, N.Y. 1966.*

7. *Stark, P.A., "Introduction to Numerical Methods", Macmillan, London, 1970.*

8. *Aziz, A. and El-Ariny, A.S., "Heat Transfer in Plane Couette Flow with Pressure Gradient and Viscous Dissipation", Journal of Engineering Sciences, Vol. 1, No. 2, 1975, pp. 69-71.*

9. *Tien, C., "Laminar Heat Transfer of Power-Law non-Newtonian Fluid — The Extension of Graetz-Nusselt Problem". The Canadian Journal of Chemical Engineering, June, 1962, pp. 130-134.*

10. *El-Ariny, A.S. and Aziz, A., "Entrance Region Heat Transfer Analysis for a Circular Tube using Rutishauser Method", to be submitted for publication.*